

# Fast-forwarding of Hamiltonians and Exponentially Precise Measurements

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## Abstract

In the early days of quantum mechanics, it was believed that the time energy uncertainty principle (TEUP) sets a bound on the efficiency of energy measurements, relating the duration,  $\Delta t$ , of the measurement, and the accuracy error in that measurement,  $\Delta E$ , by  $\Delta E \cdot \Delta t \geq \frac{1}{2}$ . Y. Aharonov and Bohm [1] gave an example in which the principle doesn't hold; whereas Aharonov, Massar and Popescu [2] showed that under certain conditions the principle applies. Can we classify when and to what extent the TEUP is violated?

Our main result is a theorem which unravels the source of such violations of the TEUP: such violations are equivalent to the ability to *fast forward* the associated Hamiltonian, namely, to simulate (using a quantum computer) its evolution for time  $t$  using significantly less than  $t$  quantum gates. Precision measurements are thus in one to one correspondence with such fast-forwarding abilities, fundamentally linking quantum algorithms and precision measurements [3]. Our theorem is stated in terms of a modified TEUP, which we call the computational TEUP (cTEUP). In this principle the time duration ( $\Delta t$ ) is replaced by the number of quantum gates required to perform the measurement, and we argue why this is more suitable to study if one is to understand the totality of physical resources required to perform an accurate measurement.

The inspiration to our result is an intriguing example we provide: a family of Hamiltonians, based on Shor's algorithm [4], which *exponentially* violates the cTEUP (as well as the TEUP), and which allow exponential fast forwarding. We further show that commuting local Hamiltonians, as well as any quadratic Hamiltonian of fermions on  $n$  sites, can be fast forwarded. An important special case is Anderson localization. The work raises the question of finding a good physical criterion for fast forwarding; in particular, can many body localization systems be fast forwarded? We rule out a general fast-forwarding method for all physically realizable Hamiltonians (unless  $\text{BQP} = \text{PSPACE}$ ). Connections to quantum metrology and to Susskind's complexification notion related to a wormhole's length are discussed.

## 1 Introduction

In quantum mechanics, position and momentum are conjugate variables, and it is well known that one can prove the position-momentum uncertainty principle  $\Delta x \cdot \Delta p \geq \frac{1}{2}$  from the properties of the Fourier transform [5]. Like position and momentum, frequency and time are also conjugate variables, and since frequency represents energy in quantum theory, a natural question is whether time and energy also obey an uncertainty principle. Several obstacles arise immediately - time is not an operator but a scalar, hence it commutes with all quantum operators; the "time" quantity of a particle is not well defined; time doesn't evolve, and can't be a constant of motion. A common misconception of the time energy uncertainty principle (TEUP) is the following:

**TEUP Misconception** *The duration  $\Delta t$  of an energy measurement with accuracy error  $\Delta E$  is bounded from below by*

$$\Delta E \cdot \Delta t \geq \frac{1}{2}. \quad (1)$$

There are several other notions that are referred to in the literature by the name time-energy uncertainty principle, or are sometimes discussed in its context (see, e.g., the survey of Busch [6]). In the

related work section 8 we will discuss our results in light of some of those (in particular, the Heisenberg limit [7–10] and the Mandelstam-Tamm relation [11]).

The TEUP misconception, Equation 1 is indeed obeyed in many cases, such as measuring the energy of the electrons in an excited atom by detecting the emitted photon. The measurement duration depends on the lifetime of the excited state, and the lifetime is inversely proportional to the spectral linewidth which represents the uncertainty in energy (see, e.g., [2]).

However, already in 1961, Y. Aharonov and Bohm [1] gave an example for a case in which the TEUP is violated: Suppose a Hamiltonian  $H$  acts on a system  $S$ , and let  $H_{meas.}$  be the Hamiltonian coupling the system and the measurement device for a time duration  $\Delta t$ , thus measuring the energy of an eigenstate of the system  $S$  with respect to  $H$ , to within some accuracy  $\Delta E$ . One can denote the unitary evolution of the system and the measurement device by

$$U_{meas} = e^{-iH_{meas}\Delta t}. \quad (2)$$

The measurement duration  $\Delta t$  can be reduced arbitrarily if an amplified measurement Hamiltonian is used:

$$U_{meas.} = e^{-iH'_{meas.}\Delta t'}, \quad (3)$$

with  $H'_{meas.} = cH_{meas.}$  and  $\Delta t' = \Delta t/c$ , providing an arbitrarily large violation of the TEUP.

On the other hand, Y. Aharonov, Massar and Popescu [2] showed (following initial results by Childs Preskill and Rene [3]) that under the condition that *nothing is known about the Hamiltonian being measured*, such an amplification is impossible, and the TEUP does in fact hold. In other words, if the Hamiltonian is treated as a *black box*, which the experimentalist can only turn on and off for some duration of time which he can control, but knows nothing about the inner makings of this box, and in particular knows nothing about the eigenvalues and eigenvectors of the Hamiltonian, the TEUP holds. In fact, even lack of knowledge of only the eigenvalues suffices for the TEUP to hold.

Thus, in some cases the TEUP is violated, whereas in the “unknown Hamiltonian” case it is obeyed. To what extent and in which situations can the TEUP be violated in physics?

The main contribution of this paper is to connect this question to a notion which we call *fast forwarding* of Hamiltonians. Fast forwarding is the ability to simulate (using a quantum computer) the evolution of a given system governed by a certain Hamiltonian to within time  $t$ , but s.t. that the simulation takes time which is *much less than*  $t$ :

**Definition 1** (fast forwarding a Hamiltonian). A normalized Hamiltonian  $H$  ( $\|H\| = 1$ ) acting on  $n$  qubits can be  $(T(n), \alpha(n))$ -fast forwarded if for any  $t \leq T$ , there exists a quantum circuit  $\tilde{U}$  with complexity  $poly(n)$  which acts on the  $n$  qubits and on additional  $c = poly(n)$  ancilla qubits initialized to 0, s.t.

$$\left\| (e^{-iHt} \otimes \mathbb{1}_{2^c} - \tilde{U}) |\psi\rangle \otimes |0\rangle \right\| \leq \alpha \quad (4)$$

It turns out that fast forwarding occurs if and only if the TEUP can be violated (a more precise definition of this violation will be given later). The main object of this paper is to point at this interesting connection and study its implications.

We start in Section 2 where we introduce a variant of the TEUP which we call the computational TEUP (cTEUP), in which the time duration of the energy measurement is replaced by the computational complexity of the energy measurement. We explain why we believe this to be a more adequate version to study in our context (and stress that this does not weaken our results, but just places them in a more accurate context). Section 3 describes an intriguing example of an exponential violation of the cTEUP based on Shor’s algorithm, which triggered this work. In this example one already sees the relation to fast forwarding. Section 4 states and proves the equivalence between fast forwarding and exponentially precise efficient energy measurements. We next proceed to study which Hamiltonians can be fast forwarded (or equivalently, their energy can be measured super efficiently). Section 5 shows that commuting local Hamiltonians, quadratic Hamiltonians, and Anderson localization are such systems. Section 6 proves that no general fast forwarding method exists for physical Hamiltonians, unless  $BQP = PSPACE$ . In Section 7 we explore quantum algorithms which can be associated with Hamiltonians, other than Shor’s, and argue why none seem to exhibit fast forwarding (not even quadratic); we also remark on an observation raised by this study, related to the existence of a quantum algorithmic speed up for a cyclic version of the graph automorphism problem. Section 8 discusses the many connections between this

work and other physical questions, including relations to metrology and sensing, the Heisenberg limit, and to a recent result by Susskind and Aaronson [12] regarding Susskind’s conjecture on the relation between the complexity of certain quantum states and the length of wormholes [13]. We end with some general conclusions and a few open questions in Section 9.

We believe that this work poses an important step in the fundamental question of further characterizing and understanding the conditions for high resolution measurements; Our results suggest that studying this question within the framework of quantum information processing, and in particular taking into account the ability to apply quantum computations to aid the measurement process, might enable very interesting insights. Hopefully, these insights will lead to experimental implications on precision measurements.

## 2 A computational version of the TEUP

Our first step is to suggest that a more adequate TEUP to investigate is not the above postulate (Equation 1) but a circuit complexity version of it.

To understand why such a modification is required, and what would that be, we return to the arbitrary violation of the TEUP given by [1]. As Aharonov and Bohm observed (see Equation 3) the two resources of time and energy can be interchanged in their case, without affecting the accuracy of the measurement; increasing the norm of the interacting Hamiltonian can shorten the duration of the measurement. Since the TEUP doesn’t take into account the interaction Hamiltonian’s norm, there is no “price” for the large norm in the TEUP - we thus arrive at a violation. However, if one does take the norm into account in their case, no violation is achieved; see further in this section for more details.

What if we insist that the norm is kept bounded? It turns out that time can also be traded with some other resource. Consider the following trick. Let  $U = e^{-iHt}$  be the time evolution corresponding to a Hamiltonian  $H = H_{meas}$  used by [1] as the measurement Hamiltonian, whose norm  $\|H\|$  can be arbitrary large. Writing

$$U = \sum_j e^{-iE_j t} |E_j\rangle \langle E_j|, \quad (5)$$

as the time evolution by  $H$ , for time  $t$ , we notice that because the eigenvalues of  $U$  are in the complex unit circle, we can always find an alternative Hamiltonian which generates the same time evolution, but which is of bounded norm:  $H' = \sum_j (E_j t \bmod 2\pi) |E_j\rangle \langle E_j|$ . Evolving according to  $H'$  for time  $t' = 1$  is equivalent to applying the unitary evolution  $U$ , and thus, we arrive at the same violation of the TEUP but with bounded norm and with finite time duration! Notice however, that now another resource is being heavily used: to apply  $H'$ , one needs to diagonalize the original Hamiltonian and compute its eigenvalues to extremely high precision. This could be highly demanding computationally.

What is revealed by the above discussion is that once one allows manipulations while performing the energy measurement, such as increasing the norm, or modifying the Hamiltonian in other ways to achieve an “equivalent” energy measurement, the TEUP can be easily bypassed. Nevertheless the resources invested in the measurement have not decreased but were just interchanged with others. The “correct” notion that we would like to capture in the TEUP is not the time duration but the totality of physical resources one is required to invest in a measurement. The underpinnings of the area of quantum computation (see [14]) tell us exactly what is the right quantity to look at: the *computational complexity* of the measurement, namely, the size of the quantum circuit *simulating* the process of the measurement, where size is measured by the number of two-qubit quantum gates (see Definition in [15]). This notion exactly takes into account all possible ways to apply a quantum process which results in a measurement of the energy of a state with respect to the given Hamiltonian. We thus postulate the following *computational complexity version* of the TEUP (which we denote by cTEUP):

**Postulate 1** (computational TEUP (cTEUP)). *An energy measurement of an eigenstate of a Hamiltonian  $H$ , with accuracy error  $\delta E$ , satisfies*

$$\delta E \cdot (\text{measurement's computational complexity}) \in \Omega(1). \quad (6)$$

We note that  $\delta E$  refers to the *accuracy error*, namely the difference between the correct eigenvalue and the outcome of the measurement.

We need to be slightly careful in defining the error. First, the accuracy of the outcome is only guaranteed with some probability, usually close to but not equal to 1. We refer to this probability as the *confidence*. More precisely, we use the following notation:

**Definition 2** ( $\eta$ -accuracy). An energy measurement is said to have accuracy  $\delta E$  with confidence  $\eta$  (we denote this as a measurement of  $\eta$ -accuracy  $\delta E$ ) if given an eigenstate with energy  $E$ , the measurement outcome  $E'$  satisfies

$$\Pr_{E'}(|E - E'| \leq \delta E) \geq \eta. \quad (7)$$

We usually set  $\eta = 2/3$ . Note that requiring the measurement to have accuracy error  $\delta E$  with confidence  $\eta$  is a slightly weaker requirement than the common requirement that the standard deviation is  $\delta E$ . In particular, when the standard deviation is specified, it is assumed implicitly that the expectation of the outcome is the correct value  $E$ . However, the expectation of the outcome  $E'$  of a measurement of accuracy  $\delta E$  and confidence  $2/3$ , might be arbitrarily far from  $E$ . Still, the *median* of many such measurements would be within  $\delta E$  from  $E$  with probability which approaches 1 exponentially fast in the number of repetitions (see the Confidence Amplification lemma, Lemma 2). The other direction does hold. When the expectation of the measurement is the correct value  $E$ , its standard deviation  $\text{std}(E)$  can be seen to provide an upper bound on the  $2/3$ -accuracy  $\delta E$  (a la Definition 2):  $\delta E \leq \sqrt{3}\text{std}(E)$ <sup>1</sup>.

There is one other source of error in the measurement process: the amount by which the measured state is modified, or *demolished*. Often one is interested in leaving the measured state in tact, as in non-demolition measurements [16]. To quantify this one can use any reasonable metric on quantum states, e.g. the fidelity or the trace metric [15]. For now, we assume by default that the demolition of the state is set to be polynomially small in the number of qubits in the system. As we will see later, if the demolition error is that small, the confidence parameter can be easily amplified and thus the exact choice of  $\eta$  doesn't matter to the question of whether a violation of the cTEUP is possible and to what extent<sup>2</sup>.

Armed with this modified principle, Postulate 1, we are able to clarify more adequately the true physical resources required for highly accurate and efficient measurements. We stress that using the cTEUP instead of the TEUP does not make the question easier but just more fundamental, as it is closer to what we really want to understand, which is what are the physical limitations on precision measurements. Notice that satisfying the cTEUP implies satisfying the TEUP; the other way round is not true, as in [1].

## 2.1 Completely known Hamiltonians

A first natural question is: perhaps when taking all the resources into account, the cTEUP does hold?

Indeed, the example in [1] is no longer a counterexample. This is because the increase in the norm is reflected also in an increased computational complexity of the measurement: Let  $f(n)$  be the time complexity of simulating the Hamiltonian  $H_{meas.}$  of Equation 2 for one time unit. The naive way to simulate  $cH_{meas.}$  for one time unit, in order to improve the accuracy by a factor of  $c$ , is to concatenate  $c$  copies of the circuit implementing  $e^{-iH_{meas.}}$ . This yields a total time complexity  $cf(n)$  - and the factor  $c$  cancels with the one we get for the improvement in accuracy. Thus the cTEUP holds in the [1] example.

However, it turns out that a simple counterexample to the cTEUP does exist. Here is a way to achieve an infinite violation of the cTEUP (as well as of the TEUP) using a simple Hamiltonian on  $n$

<sup>1</sup>This holds only if we assume that the values of the measurements are distributed by some smooth continuous distribution. If they are not, a slight modification of the definition is required: One needs to define  $\delta E^\dagger$  to be the supremum over all values for which  $\Pr_{E'}(|E - E'| < \delta E^\dagger) < \eta$  holds. Let  $E$  be the energy of the Hamiltonian's eigenstate  $\psi_E$ . Assuming that the expectation of the energy measurement  $\bar{E}$  coincides with  $E$ ,

$$\text{std}(E) = \sqrt{\sum_{E'} \Pr(E')(E' - \bar{E})^2} \geq \sqrt{\sum_{E': |E' - \bar{E}| > \delta E^\dagger} \Pr(E')(E' - \bar{E})^2} \geq \sqrt{\sum_{E': |E' - \bar{E}| > \delta E^\dagger} \Pr(E')(\delta E^\dagger)^2} \geq \sqrt{\frac{(\delta E^\dagger)^2}{3}} = \frac{\delta E^\dagger}{\sqrt{3}}$$

where in the last inequality we have assumed that  $\delta E^\dagger$  is the  $2/3$  accuracy, with the modified definition.

<sup>2</sup>We have not investigated the case in which large demolition error is allowed; it might be that this does enable considerably more efficient measurements.

spins (or qubits). Let

$$H = \sum_{i=0}^n \sigma_i^z. \quad (8)$$

Given an eigenstate, which is a tensor product of the eigenstates of each of the  $\sigma^z$ 's, a measurement of each of the spins in the eigenbasis of the Pauli  $\sigma^z$ , (the computational complexity of this measurement is  $O(n)$ ) reveals the eigenvalue to infinite precision, namely, with  $\delta E = 0$ . The demolition error however might be very large since most eigenstates are superposition of computational basis states. To avoid demolition altogether, an alternative measurement can be performed efficiently, using standard quantum computation tricks: Add a register of  $\log(n)$  qubits all initiated in the state 0, and apply the unitary version of the classical computation which computes  $w(i)$ , the number of 1's in the string  $i$  of the original system, and writes it down on the additional register. In other words, apply the unitary operator:

$$U|i\rangle|0^{\log n}\rangle = |i\rangle|w(i)\rangle \quad (9)$$

this can be done using  $n$  times  $\text{poly}(\log n)$  gates [15]. Now measure the right register, which gives the correct energy with  $\delta E = 0$ .

More generally, consider the  $n$  qubit Hamiltonian  $H = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i|$ , and assume that we have full knowledge of its eigenstates and eigenvalues in the following sense: the functions  $|i\rangle \mapsto |\psi_i\rangle$  and  $i \mapsto \lambda_i$  can be computed by a quantum computer in polynomial time in  $n$ . An infinite violation of the cTEUP can be achieved: One can first apply the unitary  $U = \sum_i |i\rangle \langle \psi_i|$  on the state to be measured, use the function  $i \mapsto \lambda_i$  to write the energy on an ancilla register, and measure the ancilla. Finally apply  $U^{-1}$  to derive the original state again without any deviation.

These infinite violations assume full knowledge of the eigenstates and eigenvalues of the Hamiltonian in the above sense.

## 2.2 Hamiltonians with unknown Eigenvalues

It turns out, that when nothing is known about the Hamiltonian, or even just about its eigenvalues, the cTEUP holds. The results of [2] show this for completely unknown Hamiltonians, namely, Hamiltonians which are given as black boxes. In fact, the proof applies as is also for Hamiltonians whose eigenstates are known but their eigenvalues are not. Theorem 8 in Appendix A provides the exact statement and proof of the TEUP for Hamiltonians with unknown eigenvalues [2], for completeness; it is slightly adapted to work in our terminology of Definition 2 rather than in the mean deviation terminology of [2]. It is straight forward to argue that this theorem implies that also the cTEUP holds for Hamiltonians with unknown eigenvalues.

**Theorem 1** (cTEUP for unknown Hamiltonians). *Let  $H$  be a Hamiltonian whose eigenvalues are unknown (namely,  $H$  is taken from a set of Hamiltonians all of which have the same set of eigenvectors, but we know nothing about their eigenvalues). Let  $G(H)$  be a quantum circuit which applies the unitary  $e^{-iH}$ , given as a black box. Let  $C(n)$  denote the computational complexity of a quantum circuit which, when given an input eigenstate of  $H$ , and which has access to  $H$  only through the black box  $G(H)$ , performs an energy measurement of the input state with respect to  $H$  with accuracy  $\delta E$  and confidence  $2/3$ . Then  $C(n)$  satisfies:*

$$\delta E \cdot C(n) \in \Omega(1). \quad (10)$$

*Proof.* The proof follows trivially from Theorem 8, which gives a lower bound for the total time duration  $\Delta t$  that the Hamiltonian  $H$  is applied to achieve such an accurate energy measurement, while taking into account the possibility of applying  $H$  in intervals and on different parts of the system. Since the access to  $H$  is only by using the circuit  $G(H)$ , which applied  $H$  for one time unit, the number of instances of  $G(H)$  being used is  $\Omega(\Delta t)$ . Hence  $C(n) \in \Omega(\Delta t) = \Omega(1/\delta E)$  by Theorem 8.  $\square$

Given the two extreme cases described in the above two subsections, of completely known Hamiltonians versus Hamiltonians where no information on the eigenvalues is given, we are confronted with the question: which Hamiltonians of the more common type, that are neither fully known nor fully unknown, allow violating the cTEUP? And to what extent?

### 3 An Exponential violation of the cTEUP based on Shor's algorithm

We describe an enlightening example, based on Shor's polynomial time quantum algorithm [4] for finding the prime factors of a given integer, which exhibits an exponential speed-up over all known classical algorithms for the same task. This algorithm gives rise to a family of Hamiltonians whose eigenvalues and eigenvectors are *not* known to us in advance, (computationally) and yet these Hamiltonians constitute a counterexample to the cTEUP (postulate 1); moreover, they violate it *exponentially*.

To define the Hamiltonians, we first recall the essential steps in the algorithm. It factors an  $n$ -bit number  $N$  by finding the order  $r$  of a randomly chosen  $y$  co-prime to  $N$  (i.e.,  $\gcd(y, N) = 1$ ), namely the period of the sequence  $y^0, y^1, y^2, \dots$ . (If one knows how to find such an  $r$ , the factors of  $N$  can then be found using an efficient classical algorithmic procedure [15]). To find the order  $r$ , the algorithm uses the unitary  $U_y$  acting on  $n$  bit strings, defined as the application of multiplication by  $y$  modulo  $N$ , where as usual we identify numbers with their binary representation, namely strings of  $n$  bits:

$$U_y |x\rangle = \begin{cases} |x \cdot y \bmod N\rangle & 0 \leq x < N \\ |x\rangle & \text{otherwise} \end{cases} \quad (11)$$

Our candidate Hamiltonian for violating the cTEUP is defined as

$$H_y = U_y + U_y^\dagger. \quad (12)$$

$H_y$  is 2-sparse row-computable (namely, has only two non-zero entries in each row and column, and given the index of the row, the two non-zero entries can be efficiently computable) and thus it can be efficiently simulated by a quantum computer (see [17, 18]).

To achieve efficient and exponentially accurate measurement of the eigenvalues of  $H_y$ , we use the fact that  $H_y$  shares the same eigenvectors with  $U_y$ , and their eigenvalues are related in a simple way. To see this, recall the orbit-stabilizer theorem [19] which implies that  $U_y$  partitions the set  $\{0, 1, \dots, N-1\}$  into orbits, each one being the orbit of some representative element in the set, and that the size of each orbit divides  $r$ . Denote  $x_\ell$  the representative of the  $\ell^{\text{th}}$  orbit,  $\mathcal{O}(x_\ell)$ . Then the eigenstates of  $U_y$  (and of  $H_y$ ) are of the form

$$|\psi_{\ell, k_\ell}\rangle = \sum_{j=0}^{|\mathcal{O}(x_\ell)|-1} e^{\frac{2\pi i j k_\ell}{|\mathcal{O}(x_\ell)|}} |x_\ell \cdot y^j \bmod N\rangle \quad \begin{matrix} \ell \in \{1, 2, \dots, \# \text{orbits}\} \\ k_\ell \in \{0, 1, \dots, |\mathcal{O}(x_\ell)|-1\} \end{matrix} \quad (13)$$

The eigenvalue of  $|\psi_{\ell, k_\ell}\rangle$  with respect to  $U_y$  is  $e^{i\varphi} = e^{\frac{2\pi i k_\ell}{|\mathcal{O}(x_\ell)|}}$ . The eigenvalue with respect to  $H_y$  is  $E_{\ell, k_\ell} = 2\cos(\varphi)$ . Estimating  $E_{\ell, k_\ell}$  to within exponential accuracy will be achieved by a measurement of  $\varphi$  to within exponential accuracy. Note that in physics,  $H_y$  describes a tight binding model of several disjoint 1D lattices, each of them with periodical boundary conditions.

The estimation of the eigenvalues of  $H_y$  is thus a standard exercise in quantum computation; for completeness we include the details below. Importantly, we notice that though it might seem that the eigenvectors and eigenvalues of the Hamiltonian are known here in advance, in fact they are not - because they depend on  $r$ , which is computationally not known. This will prove that these Hamiltonians exhibit an exponential violation of the cTEUP, postulate 1:

**Theorem 2.** Consider  $H_y$  for all  $N$  and  $y$  such that  $\gcd(y, N) = 1$ . Let us denote them by  $\{H_{n,y}\}_{n=1}^\infty$ , such that  $H_{n,y}$  acts on  $\text{poly}(n)$  many quantum bits. There exists an energy measurement procedure which given any eigenstate of  $H_{n,y}$  has  $2/3$ -accuracy  $\delta E$  such that:

$$\delta E \cdot (\text{measurement's computational complexity}) \approx 1/\exp(n). \quad (14)$$

The measurement procedure is such that the given eigenstate remains in tact.

*Proof.* We are given  $y$  and  $n$ , and a quantum register containing an eigenstate  $|\varphi\rangle$  of  $H_y$  with an unknown eigenvalue  $2\cos(\varphi)$ . Let us fix an accuracy parameter  $q = \text{poly}(n)$ . We devise a procedure to estimate  $\varphi$  to within  $2\pi \cdot 2^{-q}$  with  $2/3$  probability, using only polynomially in  $q$  (and thus in  $n$ ) quantum gates. This will prove the theorem.



Since the eigenvector is also an eigenvector of  $U_y$ , of eigenvalue  $e^{i\varphi}$ , we can use phase estimation (see section 5.2 in [15]) with respect to  $U_y$  to estimate  $\varphi$ , which is directly related to the eigenvalue  $2 \cos(\varphi)$  we need.

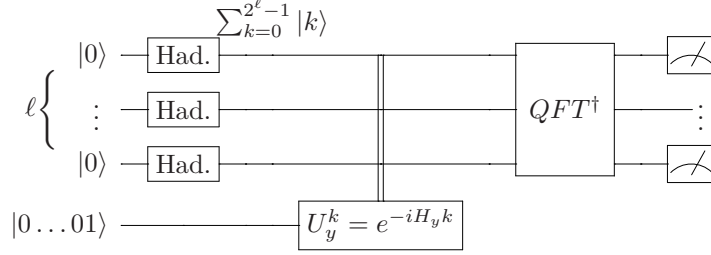


Figure 1:  $\ell$ -qubits phase estimation procedure.

The first step in phase estimation is to prepare an ancilla register in a superposition of values  $0 \dots L-1$ , where  $L = 2^\ell$ , using  $\ell$  Hadamard gates (see figure 1). We choose  $\ell = q + 6$ . Then,  $U_y^k$  is applied, conditioned that the value of the  $\ell$  control bits is  $k$  in binary representation. Finally, the inverse of the quantum Fourier transform over  $\mathbb{Z}_L$  is applied:

$$|k\rangle \xrightarrow{QFT^\dagger} \sum_{j=0}^{L-1} e^{-\frac{2\pi i j k}{L}} |j\rangle \quad (15)$$

This gives us the following sequence of implications:

$$\sum_{k=0}^{L-1} |\psi\rangle |k\rangle \rightarrow \sum_{k=0}^{L-1} U_y^k |\psi\rangle |k\rangle = \sum_{k=0}^{L-1} e^{ik\varphi} |\psi\rangle |k\rangle \rightarrow |\psi\rangle \sum_{j=0}^{L-1} |j\rangle \sum_{k=0}^{L-1} e^{i(k\varphi - \frac{2\pi j k}{L})}. \quad (16)$$

Then the first  $\ell$  qubits are measured and let  $m$  be the  $\ell$  bit outcome. Then the output estimation of  $\varphi$  is  $\varphi' = \frac{2\pi m}{2^\ell}$ .

The following lemma is useful for evaluating the errors of phase estimation.

**Lemma 1** (phase estimation confidence (adapted from 5.2.1 in [15])). Let  $U$  be a unitary and  $e^{i\varphi}$  an eigenvalue of  $U$  and let an eigenvector with this eigenvalue be given as input to the phase estimation procedure. Let  $m$  be the measurement outcome of an  $\ell$ -qubits phase estimation circuit (see figure 1). For any  $b + 1 < \ell$ ,

$$\Pr\left(\left|\varphi - \frac{2\pi m}{2^\ell}\right| > \frac{2\pi}{2^b}\right) \leq \frac{1}{2(2^{\ell-b} - 2)} \leq \frac{1}{2^{\ell-b}}. \quad (17)$$

Lemma 1 shows in our case, setting  $b = q + 4$ , the probability the estimation  $\varphi'$  is  $\pi \cdot 2^{-q-3}$  far from the value of  $\varphi$  is  $\leq 1/4$ . Since  $E = 2 \cos \varphi$ , an accuracy  $\delta\varphi$  in phase translates to accuracy  $\delta E \leq 2\delta\varphi$ . Hence the energy measurement outcome  $E' = 2 \cos \varphi'$  is  $2^{-q}$ -far from the correct value  $E$  with probability at least  $2/3$ , as required.

The time complexity of the phase estimation is  $O(n^3)$ , which consists of preparing the superposition of the ancilla register ( $O(n)$ ) quantum Fourier transform ( $O(n^2)$ ) and the controlled application of  $U^k$ . The latter is done using the method of modular exponentiation. To compute  $U_y^k$  one needs to multiply a given integer by  $y$  raised to the power  $k$  modulo  $N$ . To this end, the sequence  $y^{2^1}, y^{2^2}, \dots, y^{2^\ell} \bmod N$  is calculated classically using repeated squaring ( $O(n)$  multiplications of integers written on  $O(n)$  size registers yields  $O(n^3)$  2-gate operations), and then  $O(n)$  ctrl- $U_{y^k}$  gates are applied for a total time complexity of  $O(n^3)$ . Shor [20] improved the complexity to  $O(n^2 \log n \log \log n)$  by using improved integer multiplication techniques.  $\square$

We remark that in fact, a much simpler algorithm to measure the energy exists, more along the lines of Kitaev's original presentation of phase estimation [21]. Given an eigenstate  $\psi_E$ , then by adding a control qubit in the state  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  in another register, and applying  $U^t$  conditioned on the control qubit being 1, the state of the control qubit becomes:

$$\frac{1}{\sqrt{2}}(|0\rangle + e^{-iEt}|1\rangle) \quad (18)$$

For each  $t \in \{2^k \pi | k = 0, 1, \dots, \text{poly}(n)\}$  we can generate polynomially many control qubits in this state. Measuring these qubits (for a fixed  $k$ ) in the  $\{|+\rangle, |-\rangle\}$  basis enables evaluating one bit of the energy, with exponentially good confidence in a similar way to evaluating the bias of a classical coin by taking the average over many trials<sup>3</sup>.

This latter approach in fact makes the connection of energy measurement and fast forwarding, to be formalized in the next section for general Hamiltonians, much more apparent because the efficiency of applying  $U^t$  directly translates to energy accuracy.

## 4 Fast forwarding Hamiltonians and precision measurements

We would now like to identify the source of the ability to violate the cTEUP (postulate 1). A careful examination of the proof of Theorem 2 shows that the super-efficient energy measurement of the Hamiltonians  $H_y$  is tightly connected to the ability to raise a number  $y$  to an exponential power (modulo  $N$ , the number to be factorized), in polynomial time, i.e., to apply  $U_y^{2^n}$  in polynomial time. The straight forward way to do this would be to apply  $U_y$   $2^n$  times sequentially. Instead, this is done in polynomial time using the unitary version of a classical procedure called *modular exponentiation*. Equivalently, the time complexity for simulating the evolution of the quantum system according to the Hamiltonian  $H'_y$  s.t.  $U_y = e^{iH'_y t}$ , for time  $t = 2^n$ , can be done in time polynomial in  $n$ . We view this as *fast forwarding* of the evolution governed by the Hamiltonian  $H'_y$ , as in Definition 1. It turns out (as follows from the results of this section) that one can also achieve fast forwarding of the Hamiltonian  $H_y$ . Moreover, this is a general fact: fast forwarding the evolution governed by a given Hamiltonian, and exponentially accurate energy measurements with respect to this Hamiltonian, are equivalent.

We first provide exact definitions of the energy measurement's parameters. The definition of fast forwarding is provided in the introduction.

**Definition 3** (SEEM). A normalized Hamiltonian  $H$  ( $\|H\| = 1$ ) acting on  $n$  qubits is  $(\eta, \delta E, \beta)$ -SEEM (super-efficient energy measurable) if there exist two unitaries  $U_{SEEM}, \tilde{U}_{SEEM}$ , acting on the  $n$  qubits and on additional output/work qubits s.t.

1.  $U_{SEEM}$  is a pre-measurement with  $\eta$ -accuracy  $\delta E$  which doesn't perturb eigenstates of  $H$ , namely,

$$U_{SEEM} |\psi_E, 0, 0\rangle = |\psi_E\rangle \sum_{E'} a_{E'} |E', g(E')\rangle \quad (19)$$

where  $|\psi_E\rangle$  is an eigenvector of  $H$  with eigenvalue  $E$ ,  $E'$  is the measurement outcome and  $g(E')$  is some garbage left in the work register, and Equation 7 is satisfied.

2. The complexity of implementing  $\tilde{U}_{SEEM}$  is polynomial in  $n$  and

$$\|U_{SEEM} - \tilde{U}_{SEEM}\| \leq \beta \quad (20)$$

We note that in fact, to be completely rigorous, the above definitions should consider a family of Hamiltonians  $\{H_n\}_{n=1}^\infty$  rather than a given Hamiltonian. This will be clear from the context and is thus left implicit here and in the remainder of the paper.

**Theorem 3. [Main]** For  $n$  the number of qubits, the following two sets of Hamiltonians are equivalent:

1.  $\text{FF}_{exp}$ : A normalized Hamiltonian  $H$  acting on  $n$  qubits is in  $\text{FF}_{exp}$  if there exists an exponentially growing function  $T = O(2^{\Omega(n)})$  s.t.  $H$  is  $(T, \alpha)$ -FF for any  $\alpha = 1/\text{poly}(n)$ .

---

<sup>3</sup>Some care is needed here to deal with the cases when the value which we evaluate lies exactly or extremely close to the borderline between two possibilities, namely the bit evaluated being equally close to 0 or 1. For that matter the energy is evaluated in ternary rather than binary representation. One then evaluates the trits in this representation; For every  $i$ , at least one out of the three values the  $i^{\text{th}}$  trit can take can be ruled out with exponentially high confidence. We do not elaborate on this here.



2.  $\text{SEEM}_{\text{exp}}$ : A normalized Hamiltonian  $H$  acting on  $n$  qubits is in  $\text{SEEM}_{\text{exp}}$  if there exists a function  $\delta E = 2^{-\Omega(n)}$  s.t.  $H$  is  $(\eta, \delta E, \beta)$ -SEEM for any  $\eta, \beta = 1/\text{poly}(n)$ .

To prove the theorem, we will use two tools: the first gives efficient exponential confidence amplification of a low-demolition energy measurement, without increasing the demolition parameter  $\beta$  too much.

**Lemma 2** (Confidence amplification). Let  $\eta > \frac{1}{2}$ , and let  $H$  be a Hamiltonian on  $n$  qubits,  $\|H\| \leq 1$ , which is  $(\eta, \delta E, \beta)$ -SEEM. Then for any integer  $m \geq 1$ ,  $H$  is also  $(1 - e^{-\frac{m}{2}(1 - \frac{1}{2\eta})^2}, \delta E, m\beta)$ -SEEM.

*Proof.* Consider  $m$  applications of the non perturbing energy premeasurement unitary circuit  $U_{\text{SEEM}}$  with  $\eta$ -accuracy  $\delta E$ . The probability that the majority of these outputs are within  $\delta E$  of the correct value can be bounded by the Chernoff bound

$$\Pr(\text{majority of measurements outside the window } \delta E) \leq e^{-\frac{m}{2}(1 - \frac{1}{2\eta})^2} \quad (21)$$

Hence a median of the measurements is at distance  $\leq \delta E$  from the correct energy value with confidence  $1 - e^{-\frac{m}{2}(1 - \frac{1}{2\eta})^2}$ . We define the new premeasurement circuit  $V_{\text{SEEM}}$  to first apply  $U_{\text{SEEM}}$   $m$  different times, each time using a new ancilla register. Each such circuit writes  $E'$  on its ancilla register.  $V_{\text{SEEM}}$  then unitarily computes the median of these  $m$  outputs on an extra register. We know that had one of these outputs been measured, the probability that it is within  $\delta E$  from the correct value  $E$  is at least  $\eta$ . Since the measurements of those values mutually commute, are independent, and commute with the measurement of the median, we see that the median is within  $\delta E$  from  $E$  with probability at least  $1 - e^{-\frac{m}{2}(1 - \frac{1}{2\eta})^2}$ .

$\tilde{V}_{\text{SEEM}}$  is defined by replacing  $U_{\text{SEEM}}$   $m$  by  $\tilde{U}_{\text{SEEM}}$  in the above procedure. Since this is done  $m$  times we have

$$\|\tilde{V}_{\text{SEEM}} - V_{\text{SEEM}}\| \leq m\|\tilde{U}_{\text{SEEM}} - U_{\text{SEEM}}\| \leq m\beta. \quad (22)$$

□

The second tool allows increasing the  $T$  parameter of fast forwarding at the cost of degrading  $\alpha$ .

**Lemma 3** (FF by concatenation). For any integer  $\kappa > 0$ , if a Hamiltonian is  $(T, \alpha)$ -FF, it is also  $(T\kappa, \alpha\kappa)$ -FF.

*Proof.* The proof is by concatenation of  $\kappa$  instances of the fast-forwarding circuit; the bound of  $\alpha\kappa$  is derived by a standard telescopic argument. □

To prove Theorem 3 we start by proving that fast forwarding implies super efficient energy measurements. After this we prove the other direction.

**Claim 1.** For  $T = O(2^{\text{poly}(n)})$ , if a normalized Hamiltonian on  $n$  qubits is  $(T, \alpha)$ -FF, it is additionally  $(1 - e^{-n/18}, \frac{1}{T}, 16n\alpha \log(32T))$ -SEEM.

*Proof.* We start by using the concatenation lemma (Lemma 3) to claim the Hamiltonian is  $(16T, 16\alpha)$ -FF. Next we show that  $(16T, 16\alpha)$ -FF and  $T = O(2^{\text{poly}(n)}) \Rightarrow (\frac{3}{4}, \frac{1}{T}, 16\alpha \log(32T))$ -SEEM. The result then follows from the amplification lemma, Lemma 2 with  $m = n$ .

We use the assumption that fast forwarding of  $H$  is possible, to efficiently apply phase estimation with respect to the unitary  $V = \exp(i(H + \mathbb{1}))$ .  $V$  and  $H$  of course share eigenvectors, and an eigenvalue  $E$  of  $H$  corresponds to an eigenvalue  $e^{i\varphi}$  for  $V$  for  $\varphi = E + 1$  (recall that  $\|H\| \leq 1$  so  $0 \leq \varphi = E + 1 \leq 2 \leq 2\pi$ ).

Fix  $\ell = \lfloor \log(32T) \rfloor$  to be the number of bits of  $\varphi$  estimated in the phase estimation procedure. The procedure requires conditional applications of  $\left\{ V^{2^k} \right\}_{k=0}^{\ell-1}$ ; This is done by implementing  $\ell$  different instances of fast forwarding of  $H$ ,  $e^{iHt}$ , with  $t = 2^0, 2^1 \dots 2^{\ell-1} \leq 16T$ . Using Lemma 1, we get that the  $\ell$ -bit phase estimation procedure estimates  $\varphi$  to within  $\delta\varphi = \pi \cdot 2^{-(\ell-3)}$  with confidence  $3/4$ . We get that the procedure provides an outcome which is within  $\delta E = \delta\varphi = 4\pi \cdot 2^{-(\ell-1)} \leq \frac{4\pi}{16T} < \frac{1}{T}$  from  $E$  with confidence  $3/4$ .

To apply the  $\ell$  instances of conditional applications of powers of  $V$ ;  $\{V^{2^k}\}_{k=0}^{\ell-1}$ , we apply  $\ell$  different  $16\alpha$ -approximations of  $e^{iHt} \otimes \mathbb{1}_{2c}$  (using the fast forwarding) where each such application works on the state plus its own ancilla register initialized to 0 (as in Definition 1). We get that  $\beta \leq 16\alpha\ell \leq 16\alpha \log(32T)$ . □

**Corollary 1.**  $\text{FF}_{exp} \subseteq \text{SEEM}_{exp}$

*Proof.* A Hamiltonian  $H \in \text{FF}_{exp}$ , can be FF for some  $T = 2^{\text{poly}(n)}$ , with  $\alpha = \frac{\beta}{16n \log(32T)} = O\left(\frac{1}{\text{poly}(n)}\right)$  for any  $\beta = O\left(\frac{1}{\text{poly}(n)}\right)$ . Hence, by Claim 1,  $H$  is  $(1 - e^{-n/18}, 1/T, \beta)$ -SEEM, and therefore it is  $(\eta, 1/T, \beta)$ -SEEM for any  $\eta, \beta = O\left(\frac{1}{\text{poly}(n)}\right)$ . We conclude that  $H \in \text{SEEM}_{exp}$ .  $\square$

**Claim 2.** Let  $H$  be an  $n$  qubit Hamiltonian with  $\|H\| \leq 1$  which is  $(\eta, \delta E, \beta)$ -SEEM for  $\eta > 1/2$ . Let  $T\delta E < \frac{\pi}{2}$ , then  $H$  is also  $(T, 2\eta \sin(\delta E T) + 2(1 - \eta + \beta))$ -FF.

*Proof.* The idea of the proof is to apply the unitary  $\tilde{U}_{SEEM}$  approximating the premeasurement of the energy, which exists since the Hamiltonian can be super-efficiently measured, by Definition 3. Then, based on the output  $E'$  of this premeasurement, written on the quantum register, multiply the state by the phase  $e^{-iE't}$  (denote this by the gate  $V$ ), and finally apply the inverse of the approximated premeasurement unitary. Let  $|\alpha\rangle = |\psi_E\rangle \otimes |0\rangle$ . First we consider the exact premeasurement with no demolition ( $\beta = 0$ ),  $U_{SEEM}$ ; notice that it commutes with  $H$ :

$$\begin{aligned} \left\| \left( U_{SEEM}^\dagger V U_{SEEM} - e^{-iHt} \otimes \mathbb{1}_W \right) |\alpha\rangle \right\| &= \left\| \left( U_{SEEM}^\dagger V U_{SEEM} - U_{SEEM}^\dagger (e^{-iHt} \otimes \mathbb{1}_W) U_{SEEM} \right) |\alpha\rangle \right\| \\ &= \left\| (V U_{SEEM} - (e^{-iHt} \otimes \mathbb{1}_W) U_{SEEM}) |\alpha\rangle \right\|, \end{aligned} \quad (23)$$

where the Hilbert space of the work/output register is denoted by  $W$ . On a specific eigenvector  $\psi_E$ :

$$\begin{aligned} \left\| V U_{SEEM} |\psi_E, 0, 0\rangle - (e^{-iHt} \otimes \mathbb{1}_W) U_{SEEM} |\psi_E, 0\rangle \right\| &= \left\| \sum_{E'} a_{E'} (e^{-iE't} - e^{-iEt}) |\psi_E, E', g(E')\rangle \right\| \\ &= \left\| \sum_{E': |E'-E| \leq \delta E} a_{E'} (e^{-iE't} - e^{-iEt}) |\psi_E, E', g(E')\rangle + \sum_{E': |E'-E| > \delta E} a_{E'} (e^{-iE't} - e^{-iEt}) |\psi_E, E', g(E')\rangle \right\| \\ &\leq 2\eta \sin(\delta E t) + 2(1 - \eta), \end{aligned} \quad (24)$$

where the last inequality is correct for  $t \leq \pi/2\delta E$ . Notice that the above holds for any state  $|\psi\rangle = \sum_{c_E} c_E |\psi_E\rangle$ , using the fact that both  $U_{SEEM}$  and  $V$  leave the left register in tact. The proof follows since we have  $\left\| \tilde{U}_{SEEM}^\dagger V \tilde{U}_{SEEM} - U_{SEEM}^\dagger V U_{SEEM} \right\| \leq 2\beta$ .  $\square$

**Corollary 2.** Let  $H$  be a normalized Hamiltonian on  $n$  qubits, which is  $(\eta, \delta E, \beta)$ -SEEM for  $\eta > 1/2$  and  $\beta < \pi/2$ . Then  $H$  is also  $(\beta/\delta E, 2n\beta + 2^{-O(\text{poly}(n))})$ -FF.

*Proof.* Using lemma 2 with  $m = n - 1$  we reach an  $(1 - e^{-\frac{(n-1)\eta}{2}(1 - \frac{1}{2\eta})^2}, \delta E, (n-1)\beta)$ -SEEM. Now choose  $T = \beta/\delta E$  and since  $T\delta E = \beta < \pi/2$  we can apply claim 2. The FF error  $\alpha$  according to Claim 2 is bounded by  $2n\beta + 2^{-\text{poly}(n)}$ .  $\square$

**Corollary 3.**  $\text{SEEM}_{exp} \subseteq \text{FF}_{exp}$

*Proof.* Let  $H \in \text{SEEM}_{exp}$ , with some inverse exponential function  $\delta E$ . We choose  $T$  to be any exponentially growing function such that  $T\delta E$  decays faster than any polynomial (say,  $T = \frac{1}{\delta E^{0.99}}$ ). Let  $\alpha = O\left(\frac{1}{\text{poly}(n)}\right)$  be a goal parameter for the fast forwarding. By assumption,  $H$  is  $(\eta = 2/3, \delta E, \beta = \alpha/3n = O\left(\frac{1}{\text{poly}(n)}\right))$ -SEEM. From corollary 2,  $H$  is also  $(\beta/\delta E, \alpha)$ -FF. By our choice of  $T$ , it is thus  $(T, \alpha)$ -FF. Since this holds for any inverse polynomial  $\alpha$ , we have  $H \in \text{SEEM}_{exp}$ .  $\square$

We remark that one can prove similar equivalences between SEEM and FF for other behaviors of  $T$  and  $\delta E$ , not necessarily those which grow exponentially with  $n$ . However, there seems to be some inherent (constant) loss in parameters when moving between the notion of FF to within time  $T$ , and SEEM to within accuracy  $1/T$ ; which is why the equivalence (Theorem 3) is stated in terms of exponential functions rather than in terms of exact parameters.

## 5 Exponential Fast Forwarding for Physical Hamiltonians

### 5.1 Commuting local Hamiltonians

A class of Hamiltonians that can be easily fast forwarded are commuting local Hamiltonians. A Hamiltonian  $H$  is a commuting  $k$ -local Hamiltonian if it is of the form

$$H = \sum_j H_j, \quad (25)$$

where every term  $H_j$  acts non-trivially on at most on  $k$  qubits, and  $[H_i, H_j] = 0$  for all  $i, j$ .

**Theorem 4.** *If  $H$  is an  $n$  qubit normalized commuting  $k$ -local Hamiltonian, with  $k = O(\log(n))$ , then it can be  $(T, \alpha)$ -fast forwarded with  $T = 2^{\Omega(n)}$  and arbitrary exponentially small  $\alpha$ .*

*Proof.* Since  $H_j$  commute, we have

$$e^{-iHt} = \prod_j e^{-iH_j t}. \quad (26)$$

It thus suffices to be able to implement  $e^{-iH_j t}$  for  $t$  exponentially large, with an appropriate exponentially small accuracy. Given the description of  $H_j$ , let  $U_j$  be the matrix which diagonalizes it,

$$D_j = U_j H_j U_j^\dagger \quad (27)$$

All entries of  $U_j$  can be efficiently calculated classically to within exponential accuracy, since the dimension of the matrix is  $2^k \times 2^k$  for  $k = O(\log(n))$ ; its associated eigenvectors  $|\psi_m\rangle$  and most importantly the corresponding eigenvalues  $\lambda_m$  can also be calculated classically with exponential accuracy. This follows from classical results about matrix calculations, see e.g., [22, 23]. Since the dimension of  $U_j$  is polynomial, and all its entries are known,  $U_j$  can be applied efficiently by a quantum computer, on the appropriate  $k$  qubits, to achieve the transformation  $|\psi_m\rangle \mapsto |m\rangle$ , for  $H_j$ . Then using a quantum computer one can apply the transformation

$$|m\rangle \mapsto e^{-i\lambda_m t} |m\rangle \quad (28)$$

to within exponentially good accuracy; and then the quantum computer can apply  $U_j^\dagger$ , which all together achieves  $e^{-iH_j t}$  to within exponentially good accuracy.  $\square$

The toric code Hamiltonian constitutes [24] an important example for a commuting 4-local Hamiltonian; We note that unlike what one might suspect, the time evolution of commuting local Hamiltonians generates very interesting behavior from the computational perspective; even evolving for a unit time results in generating distributions which are hard to even approximately simulate classically (under commonly believed computational assumptions). See [25, 26].

We stress that the important part of Theorem 4 is the fact that fast forwarding is possible for an *exponentially long time*; The fact that exponentially good accuracy  $\alpha$  is achievable is not important, since this is a result of assuming ideal quantum computation. If one takes into account noise and decoherence which will necessarily appear in practical applications, and thus augments the above theorem with quantum fault tolerance architecture (see e.g. [24, 27–29]) then the exponential fast forwarding abilities will be maintained, but the  $\alpha$  will be degraded to inverse polynomial.

### 5.2 Fast forwarding quadratic Hamiltonians

Using a similar idea to the above, one can derive exponential fast forwarding for a wide class of physically interesting Hamiltonians, called quadratic Hamiltonians. An important special case of this class is Anderson's model for electron localization [30].

We provide this here for the case of Fermions. We presume a quadratic Hamiltonian of bosons can also be fast forwarded, but haven't checked it. The states of  $n$  indistinguishable Fermions distributed over  $m = \text{poly}(n)$  modes are described by Fock space [31] of possibly exponential dimension in  $n$ :  $\binom{m}{n}$  in case of fermions, or  $\binom{n+m-1}{m-1}$  in case of bosons. A quadratic Hamiltonian is defined as follows:

$$H = \sum_{i,j}^m A_{i,j} a_i^\dagger a_j + \frac{1}{2} \sum_{i,j} B_{i,j} a_i a_j + \frac{1}{2} \sum_{i,j} B_{j,i}^* a_i^\dagger a_j^\dagger \quad A = A^\dagger, B = B^\dagger \quad (29)$$

The Hermiticity of  $H$  follows from the fact that the matrices  $A$  and  $B$  are Hermitian.  $a_i, a_i^\dagger$  are the creation and annihilation operators, satisfying the anti commutation relations for fermions,

$$\{a_i, a_j\} = 0 \quad \{a_i^\dagger, a_j^\dagger\} = 0 \quad \{a_i, a_j^\dagger\} = \delta_{i,j}. \quad (30)$$

$a_i^\dagger a_j^\dagger = -a_j^\dagger a_i^\dagger$  hence equation 29 takes the form

$$H = \sum_{i,j} A_{i,j} a_i^\dagger a_j + \frac{1}{2} \sum_{ij} B_{i,j} a_i a_j - \frac{1}{2} \sum_{i,j} B_{i,j}^* a_i^\dagger a_j^\dagger. \quad (31)$$

We now assume that we can physically implement any quadratic Hamiltonian, s.t. the error in each coefficient is at most inverse polynomial. Then:

**Theorem 5.** *Let  $H$  be a quadratic Hamiltonian of  $n$  Fermions with  $\text{poly}(n)$  modes.  $H$  can be  $(T, \alpha)$ -fast forwarded with  $T = 2^{\Omega(n)}$  and arbitrary inverse polynomial  $\alpha$ .*

*Proof.* The proof idea is to efficiently “diagonalize” the Hamiltonian by the Bogolubov transformation [32,33] to the form  $H = \sum_i \lambda_i b_i^\dagger b_i + \text{tr}(A)/2$ . The operators  $b_i, b_i^\dagger$  are called quasiparticle annihilation and creation operators respectively, and they inherit the commutation/anti commutation relations of  $a_i, a_i^\dagger$  as in Equations 30. Additionally, the number operator  $b_i^\dagger b_i$  has integer eigenvalues. Fast forwarding is enabled by efficiently evolving the system under  $H' = \sum_i (\lambda_i t \bmod 2\pi) b_i^\dagger b_i + \text{tr}(A)/2$  for one time unit.

We now describe the details, using standard claims in Physics (whose proofs can be found in the Appendix for completeness):

**Claim 3.** Let  $\mathbf{a}$  be a column vector whose  $j^{\text{th}}$  coordinate is  $a_j$  and let  $\mathbf{a}^\dagger$  be a column vector whose  $j^{\text{th}}$  coordinate is  $a_j^\dagger$ . The Hamiltonian  $H$  can be written as

$$H = \frac{1}{2} \begin{pmatrix} \overline{\mathbf{a}^\dagger} & \overline{\mathbf{a}} \end{pmatrix} \begin{pmatrix} A & B^* \\ B & -A^* \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix} + \frac{1}{2} \text{tr}(A) \quad (32)$$

Here, the overline indicates a matrix transposition, i.e.,  $\overline{\mathbf{a}^\dagger}, \overline{\mathbf{a}}$  are the row vectors corresponding to  $\mathbf{a}^\dagger, \mathbf{a}$  respectively.

**Claim 4.** The traceless part of the Hamiltonian can be diagonalized:

$$H = \frac{1}{2} \begin{pmatrix} \overline{\mathbf{a}^\dagger} & \overline{\mathbf{a}} \end{pmatrix} U D U^\dagger \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix} + \frac{1}{2} \text{tr}(A) \quad (33)$$

where  $D$  is a real diagonal matrix s.t.  $D_{j,j} = -D_{j+n,j+n}$  and  $U$  is unitary. Furthermore, there exist matrices  $V_1, V_2$  s.t.  $U$  is a block matrix in the form  $U = \begin{pmatrix} V_1 & V_2^* \\ V_2 & V_1^* \end{pmatrix}$

**Claim 5.** By defining

$$\begin{pmatrix} \mathbf{b} \\ \mathbf{b}^\dagger \end{pmatrix} = U^\dagger \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}. \quad (34)$$

The Hamiltonian takes the form

$$H = 2 \sum_{i=1}^m b_i^\dagger b_i D_{i,i} + \frac{1}{2} \text{tr}(A) \quad (35)$$

The new operators obey the anti-commutation relations of fermions. In addition, the eigenvalues of  $b_i^\dagger b_i$  either 0 or 1.

We can now use the above claims to achieve fast forwarding. In Equation 35, the modes  $b_i$  are independent, and in particular, the Hamiltonian is a sum of  $m$  commuting terms. Therefore, an evolution under  $H$  for time  $t$  can be implemented by

$$e^{iHt} = e^{-it \text{tr}(A)/2} \prod_j e^{2i D_{j,j} t b_j^\dagger b_j} \quad (36)$$

Since the eigenvalues of each term are  $D_{j,j}t$  times an integer (using Claim 5) we have that if we replace  $\mathcal{H}$  by a matrix with the same eigenvectors but with eigenvalues  $(D_{j,j}t \bmod 2\pi)$ , it will have the same evolution on any state, in other words if we define

$$H' = \frac{1}{2} \begin{pmatrix} \overline{\mathbf{a}^\dagger} & \overline{\mathbf{a}} \end{pmatrix} U D' U^\dagger \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix} + \frac{1}{2} \text{tr}(A) \quad (37)$$

$$D'_{i,j} = \delta_{i,j} D_{i,i} t \bmod 2\pi \quad (38)$$

we get

$$e^{-iH'} = e^{-iHt} \quad (39)$$

Hence it is sufficient to simulate the evolution under  $H'$  to time  $t = 1$ . To do this we observe that Equation 37 means that  $H'$  is a quadratic Hamiltonian in  $\{a_i\}$  and  $\{a_i^\dagger\}$ , whose coefficients can be calculated by a classical computer in time polynomial in  $m$  to within exponential accuracy [22,23]. Assuming that we can implement any quadratic Hamiltonian of polynomial number of coefficients exactly, we need only apply  $H'$  for one time unit to fast forward  $H$  for exponential duration  $t$ , with arbitrary exponentially small  $\alpha$ . However, the assumption that a quadratic Hamiltonian with general coefficients can be implemented exactly is not realistic; Assuming inverse polynomial error in each of the coefficients in the quadratic Hamiltonian results in an overall inverse polynomial error and thus would still lead to a fast forwarding procedure for exponential duration of time, but with inverse polynomial error  $\alpha$ .  $\square$

## 6 Impossibility of a Generic Fast Forwarding Procedure for realizable Hamiltonians

Perhaps any physically realistic Hamiltonian (one which can be efficiently simulated by a quantum circuit) can be fast-forwarded? We show that this is highly unlikely: even if a subset of the Hamiltonians - namely, the 2-sparse row-computable Hamiltonians - can be exponentially fast forwarded, the complexity class PSPACE equals BQP (which is highly unlikely).

2-sparse row-computable Hamiltonians (as mentioned in Section 3) are Hamiltonians with at most two non-zero entries per row, and such that given the row number, it is possible (either quantumly or classically) to efficiently compute the column indices and values of the non-zero entries of this row. It is known that such Hamiltonians are efficiently simulable by a quantum circuit [17,18].

**Theorem 6.** *A generic procedure for  $(T = 2^{(n^{1/c})}, \alpha = n^{-4/c})$ -fast forwarding a 2-sparse row computable Hamiltonians, with  $c > 1$ , does not exist (unless  $BQP = PSPACE$ ).*

The structure of the proof is as follows: we first assume a fast forwarding procedure with better parameters, namely we start with a  $(T = 5^n, \alpha = n^{-4})$ -FF procedure, and utilize it to design a polynomial time quantum algorithm which solves the PSPACE-complete problem OTHER END OF THIS LINE (OEOTL) [34]. We will later show that the parameters in the statement of the theorem suffice to imply the assumed parameters, essentially by a simple padding argument.

Let  $G = (V, E)$  be a directed graph where every vertex is represented by an  $n$  bits vectors ( $2^n$  vertices). The edges are represented by two polynomial size circuits  $S$  and  $P$ . There is an edge from  $u$  to  $v$  only if  $S(u) = v$  and  $P(v) = u$ ; hence  $G$  contains paths, cycles, or isolated vertices.

**Definition 4** (OEOTL). Given that the vertex  $0^n$  has no incoming edge but has an outgoing edge, find the other end of the line that starts with  $0^n$ .

Let  $A_G$  be the adjacency matrix of the undirected graph induced by  $G$ . We use  $H = \frac{1}{2}A_G$  as a 2-sparse row-computable Hamiltonian  $H$  ( $\|H\| = 1$ ). Without loss of generality, we denote the vertices along the path by  $0 \dots L-1$  with  $L \leq 2^n$ . Evolving the the state  $|0\rangle$  under  $H$  keeps the system in the subspace of all vertices in the path. The eigenvalues and eigenvectors of the Hamiltonian in this subspace are:

$$E_k = \cos\left(\frac{\pi k}{L+1}\right) \quad k = 1 \dots L \quad (40)$$

$$|\psi_k\rangle = \sqrt{\frac{2}{L+1}} \sum_{j=0}^{L-1} \sin\left(\frac{\pi k(j+1)}{L+1}\right) |j\rangle, \quad (41)$$

which is easy to check:

$$H|\psi_k\rangle = \frac{1}{2}\sqrt{\frac{2}{L+1}}\sum_{j=0}^{L-1}\left[\sin\left(\frac{\pi k j}{L+1}\right) + \sin\left(\frac{\pi k(j+2)}{L+1}\right)\right]|j\rangle = \sqrt{\frac{2}{L+1}}\sum_{j=0}^{L-1}\left[\sin\left(\frac{\pi k(j+1)}{L+1}\right)\cos\left(\frac{\pi k}{L+1}\right)\right]|j\rangle. \quad (42)$$

We assume that a generic  $(T = 5^n, \alpha = n^{-4})$ -FF is possible for any 2-sparse row computable Hamiltonian. By Claim 1 this means that SEEM for this Hamiltonian is possible with parameters  $\eta = 1 - e^{-n/18}$ ,  $\delta E = 5^{-n}$ ,  $\beta = 40n^{-2}$ . By Equation 40 at most one eigenvalue fits in a window of size  $2\delta E$ . This is because the minimal gap is between  $k = 1, 2$  (or  $k = L - 1, L$ ).  $\cos(\frac{\pi}{L+1}) - \cos(\frac{2\pi}{L+1}) = 2\sin(\frac{\pi}{2(L+1)})\sin(\frac{3\pi}{2(L+1)}) \geq \frac{3\pi^2}{8(L+1)^2} \geq \frac{2}{(L+1)^2} \geq 2\delta E$ , the leftmost inequality is due to  $\sin x > x/2$  for  $0 < x < \pi/4$ . Hence, estimating the energy of any state to within such  $\delta E$  should intuitively approximately project the state onto an eigenvector.

The polynomial time algorithm is as follows:

1. Let  $v_0 = 0$ ,  $H_0 = H$
2. For  $i=1$  to  $100n$ 
  - (a) Check that 10 steps forward from  $v_0$  the end of the line is not reached; if it is, output it and exit.
  - (b) Perform a  $(1 - e^{-n/18}, 5^{-n}, 40n^{-2})$ -SEEM on the state  $|v_{i-1}\rangle$  under  $H_{i-1}$ .
  - (c) Measure in the vertices basis, denote the result by  $v_i$ . Let  $H_i$  be the original Hamiltonian  $H$  with the edge  $(v_i - 1, v_i)$  removed.
3. Move 10 steps forward from the vertex reached. If the end of the line is found return it and exit. Otherwise, the algorithm fails.

### Proof of correctness:

The idea of the proof is to make progress on the path, as follows: Starting from the first node  $v_0$ , we use the SEEM (stage 2b) to measure the energy with respect to  $H_0$  (assume  $\beta = 0$  for now). Due to the high accuracy of the measurement, the resulting state, conditioned on the measurement outcome, is close to an eigenstate. All eigenstates are symmetric around the middle of the path, hence the measurement in the vertices basis in stage 2c, yields with good probability a vertex  $v_1$  that is closer to the end of the path than to  $v_0$  namely, the remaining path length is likely to be halved. We call this event a successful iteration, and show its probability is at least  $1/10$  if the path length is less than 10. An unsuccessful iteration does not increase the length of the path, it just doesn't succeed in shrinking by half. The vertex  $v_1$  is now the next starting point, and  $H_1$  is fixed to prevent going backwards by correcting  $H$  to not include the edge connecting  $v_1$  to the previous vertex on the line. After  $n$  successful iterations, the vertex reached should be the end of the line. By Chernoff, the probability of at least  $n$  successful iterations with  $1/10$  success probability, out of  $100n$  iterations is  $\geq 1 - e^{-81n/20}$ , which is exponentially close to one. If at some stage the length of the path is smaller than 10, the success probability may be smaller than  $1/10$ , but the end of the line is found in stage 3. An analysis of  $\beta > 0$  concludes the proof.

Let  $U_{meas.}$  be the pre-measurement with  $\beta = 0$ , and confidence  $\eta = 1 - e^{-n/18}$  with respect to  $H_i$ . Suppose the vertex found in the previous round is  $v = v_i$ , and consider applying  $U_{meas.}$  to  $|v, 0, 0\rangle$  where the additional two registers are the output and work registers. Denote the result of measuring the two additional registers by  $\varepsilon_j, g$ . Let  $f$  be the function s.t.  $\psi_{f(j)}$  is the eigenstate of  $H_i$  with energy closest to  $\varepsilon_j$  (the lower energy eigenstate if there is a tie).  $f$  is well defined since all eigenvalues of  $H_i$  within the relevant subspace have multiplicity 1. We omit adding an  $i$  index to  $f$ , and to the eigenstates/eigenvalues of  $H_i$  since both  $H$  and  $H_i$  in the relevant subspaces are Hamiltonians of paths, only the length of the path and the starting vertex change.

**Claim 6.** Let  $a_j$  be the amplitude of  $\psi_{f(j)}$  after measuring  $\varepsilon_j, g$ . The expectation of  $|a_j|^2$  over  $j, g$  satisfies:  $\mathbb{E}_{j,g}(|a_j|^2) = \sum_{g,j} |a_j|^2 \Pr(\varepsilon_j, g) \geq \eta$ .



*Proof.* Let  $f^{-1}$  be the preimage of  $f$ ,

$$\begin{aligned}\mathbb{E}_{j,g}(|a_j|^2) &= \sum_{g,j} |a_j|^2 \Pr(\varepsilon_j, g) = \sum_{g,j} \Pr(\psi_{f(j)}|\varepsilon_j, g) \Pr(\varepsilon_j, g) = \sum_{g,j} \Pr(\varepsilon_j, g, \psi_{f(j)}) \\ &= \sum_k \Pr(\psi_k) \sum_{g,j:j \in f^{-1}(k)} \Pr(\varepsilon_j, g|\psi_k) \geq \eta\end{aligned}\tag{43}$$

The last inequality, is due to the  $\eta$ -confidence of the measurement, and that all measurement outcomes in the window  $\delta E$  around  $E_k$  are in  $f^{-1}(k)$ .  $\square$

**Claim 7.** Let  $L_i = L - v_i \geq 10$  and  $\ell_i = \lceil \frac{L_i}{2} \rceil$ . The probability for a successful iteration, i.e.,  $v_{i+1} \geq v_i + \ell_i$  is at least  $1/10$  for the value of  $\eta$ .

*Proof.* After measuring  $\varepsilon_j, g$ , the state of the system is  $a_j |\psi_{f(j)}\rangle + \sqrt{1 - |a_j|^2} |\psi_{f(j)}^\perp\rangle$ . We define the  $\Pi_i$  to be a projection on the vertices  $v \geq v_i + \ell_i$ . The symmetry of the eigenstates around the middle of the path implies that  $2 \|\Pi_i |\psi_{f(j)}\rangle\|^2 + \frac{2}{L_i+1} \geq 1$ , therefore  $\frac{1}{2} - \frac{1}{L_i+1} \leq \|\Pi_i |\psi_{f(j)}\rangle\|^2 \leq \frac{1}{2}$ .

$$\begin{aligned}\Pr(v_{i+1} \geq v_i + \ell_i | \varepsilon_j, g) &= \left\| \Pi_i \left( a_j |\psi_{f(j)}\rangle + \sqrt{1 - |a_j|^2} |\psi_{f(j)}^\perp\rangle \right) \right\|^2 \\ &\geq |a_j|^2 \|\Pi_i |\psi_{f(j)}\rangle\|^2 + (1 - |a_j|^2) \|\Pi_i |\psi_{f(j)}^\perp\rangle\|^2 - 2|a_j| \sqrt{1 - |a_j|^2} \|\Pi_i |\psi_{f(j)}\rangle\| \|\Pi_i |\psi_{f(j)}^\perp\rangle\| \\ &\geq |a_j|^2 \left( \frac{1}{2} - \frac{1}{L_i+1} \right) - |a_j| \sqrt{2 - 2|a_j|^2}\end{aligned}\tag{44}$$

Using the inequality  $x\sqrt{2-2x^2} \leq 99(1-x^2) + 0.01$  for  $0 \leq x \leq 1$ , we bound the probability by  $|a_j|^2 \left( 99.5 - \frac{1}{L_i+1} \right) - 99.01$ . Finally, we average over all  $g, j$ :

$$\begin{aligned}\Pr(v_{i+1} \geq v_i + \ell_i) &\geq \sum_{g,\varepsilon_j} \Pr(v_{i+1} \geq v_i + \ell_i | \varepsilon_j, g) \Pr(\varepsilon_j, g) \geq \mathbb{E}_{j,g}(|a_j|^2) \left( 99.5 - \frac{1}{L_i+1} \right) - 99.01 \\ &\geq \eta \left( 99.5 - \frac{1}{L_i+1} \right) - 99.01\end{aligned}\tag{45}$$

With  $\eta = 1 - e^{-n/18}$ , the probability for a successful iteration is at least  $1/10$  for  $L_i > 10$ .  $\square$

**Claim 8.** When the algorithm applies the SEEM with  $\beta = 0$ , it succeeds with probability at least  $1 - e^{-81n/20}$ .

*Proof.* Consider the  $100n$  iterations in the protocol, when if the protocol had ended before completing all iterations, the iteration is simply idle. An iteration  $i$  is declared “successful” if either it is idle, or if not, the length of the path had been halved during step 2(c) of this iteration. By this definition and by Claim 7, the probability of the  $i^{\text{th}}$  iteration to be successful is  $> 1/10$ , even when we condition on what happened in previous iterations. Let  $X$  be the number of successful iterations out of the  $100n$  iterations. We want to bound the probability that  $X > n$  from below. We note that this probability is bounded from below by the corresponding probability for the number  $Y$  of successful iterations when we have  $100n$  i.i.d Bernoulli variables, each with probability exactly  $1/10$  for success. For i.i.d variables we can use the Chernoff bound,

$$\Pr(Y \leq (1 - \delta)\mu) \leq e^{-\delta^2 \mu/2}.\tag{46}$$

where  $\mu$  is the expectation of  $Y$ , and we have here  $\mu = 10n$ . Setting  $(1 - \delta)\mu = n$ ,  $\delta = 9/10$ , we have  $\Pr(Y > n) \geq 1 - e^{-81n/20}$ . This means that  $\Pr(X > n)$ , the probability for at least  $n$  successful iterations, is at least  $1 - e^{-81n/20}$ . After  $n$  successful iterations the path length must have reached below  $10$  since  $\frac{L}{2^n} \leq 1$ , and the algorithm succeeds in finding the end of the line.  $\square$

We analyze what is the success probability of the algorithm with  $\beta > 0$ . To this end, consider first a unitary version of the above algorithm, still with  $\beta = 0$ , where the only measurement is at the end. In stage 2c, we copy (using cNOTs) the result  $v_i$  to a separate register in every iteration instead of measuring. In stage 2b we apply the super efficient energy pre-measurement, where the Hamiltonian is conditioned on the copy of  $v_i$ . The outcome and the garbage are written on a separate register in every iteration. At the end of the algorithm, an indicator qubit is set to 1 if the algorithm found the end of the line and 0 otherwise. In this version the algorithm is unitary, and the only difference between  $\beta = 0$  and  $\beta > 0$  cases are the  $100n$  instances of SEEMs. Thus, at the end of the algorithm, just before measuring, the state with  $\beta > 0$  SEEMs (denoted  $\tilde{\xi}$ ) deviates at most by  $100\beta n$  from the state in which  $\beta = 0$  SEEMs were used (denoted  $\xi$ ). Let  $\Pi$  be the projection on a successful outcome of the algorithm, i.e., the indicator qubit is 1. We bound the algorithm success probability for  $\beta > 0$ :

$$\left\| \Pi |\tilde{\xi}\rangle \right\|^2 = \left\| \Pi |\xi\rangle + \Pi \left( |\tilde{\xi}\rangle - |\xi\rangle \right) \right\|^2 \geq (\|\Pi |\xi\rangle\| - 100\beta n)^2 \geq \|\Pi |\xi\rangle\|^2 - 200\beta n \quad (47)$$

Hence the success probability is reduced by  $200\beta n$ . Since  $\beta = 40n^{-2}$  the success probability is polynomially close to 1.

The main contribution to the time complexity of the algorithm is from the  $100n$  rounds of step 3. The SEEM in each round calls  $O(\log n)$  times to the fast forwarding procedure. Hence the time complexity is polynomial in  $n$ .

We conclude the proof of Theorem 6 by relaxing the demand for the generic FF procedure. Consider a generic fast forwarding procedure for  $n$  qubit Hamiltonians with parameters  $T = 2^{(n^{1/c})}$ , and  $\alpha = n^{-4/c}$  with  $c > 1$ . This procedure is weaker than a generic fast forwarding procedure with parameters  $T = 5^n$  and  $\alpha = n^{-4}$  used in the proof, however one can use the weaker procedure to  $(5^n, n^{-4})$ -FF any Hamiltonian.

Given an  $n$  qubit Hamiltonian  $H$ , one can define an  $m = (3n)^c$  qubit Hamiltonian  $H' = H \otimes \mathbb{1}_{2^{m-n}}$ .  $H'$  is still 2-sparse row computable, therefore it can be FF using the weaker procedure for  $T = 2^{(m^{1/c})} \geq 5^n$  and with  $\alpha = m^{-4/c} = (3n)^{-4} < n^{-4}$  (polynomial complexity in  $m$  is also polynomial in  $n$ ). Hence OEOTL can be solved efficiently using the weaker generic FF procedure.  $\square$

## 7 Quantum algorithms and fast-forwarding Hamiltonians

One can ask a conceptual question: is fast-forwarding Hamiltonians the true underlying source for all quantum algorithmic speed-ups? It turns out that in fact this is far from being the case. Indeed, like in Shor's algorithm, the Abelian hidden subgroup problem (HSP) is solved [21, 35] by efficiently utilizing phase estimation to exponential accuracy, thus one can associate a Hamiltonian to the problem, and the quantum algorithm can be translated to a cTEUP violation in measuring the energies with respect to this Hamiltonian. We believe (though we have not worked out the details) that this is also the case for the recent extensions of Shor's algorithm to finding unit groups of number fields [36, 37], which are also based on phase estimation of the eigenvalue of a unitary applied to exponential powers. However, to our current understanding, other than these few direct extensions of Shor's algorithm, none of the other known quantum algorithmic speed-ups can be related to fast forwarding – not even quadratic fast forwarding. We note that some of these algorithms can be viewed as an energy measurement of a corresponding Hamiltonian, as we describe below, however, the quantum speed-up does not result from a FF of this Hamiltonian. We describe this in three interesting cases.

1. **The exponential speed-up of the quantum walk on two glued binary trees [38]:** As shown in [38], the glued trees problem is highly symmetric, and the search is limited to a subspace of dimension *linear* in the number of qubits. In addition, [38] show that the spectral gap of the Hamiltonian in that subspace is inverse polynomial.

One can in fact view this process as an energy measurement, except not an accurate one. To see how continuous time quantum walks (CTQW) are related to energy measurements, consider the following analogy: In CTQW, a value  $t$  is chosen uniformly over  $[0, T]$  and the system is evolved by  $e^{-iHt}$  and then measured. Almost equivalently, one can add to the state an ancilla register, initiated in the superposition over all values of time  $\frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} |t\rangle$ , and then apply the Hamiltonian on the state for a duration  $t$  conditioned that the value in the ancilla register is  $t$ , and finally discard

the  $t$  register. This latter procedure is effectively a phase estimation (i.e., energy measurement), with the outcome traced out.

However, the algorithm in [38] only requires polynomial accuracy to perform this energy measurement, and in order to do this it simply applies the Hamiltonian for a polynomial amount of time, and does not utilize any fast-forwarding (equivalently, it does not violate the cTEUP).

2. **Grover's quadratic algorithmic speed-up [39]:** In the Grover's algorithm, an initial state  $|s\rangle$  which is a uniform superposition over a search space of size  $N$  is rotated slowly to the marked state  $\omega$ , and reaches its proximity after  $O(N^{-1/2})$  applications of the iterator  $U = (\mathbb{1} - 2|\omega\rangle\langle\omega|)(2|s\rangle\langle s| - \mathbb{1})$ .  $U$  may be written as:

$$U = (\mathbb{1} - 2|\omega\rangle\langle\omega|) \left( \frac{2}{N} \left( (N-1)|s'\rangle\langle s'| + |\omega\rangle\langle\omega| + \sqrt{N-1}(|s'\rangle\langle\omega| + |\omega\rangle\langle s'|) \right) - \mathbb{1} \right) \quad (48)$$

where  $|s\rangle = \sqrt{(N-1)/N}|s'\rangle + \sqrt{1/N}|\omega\rangle$ . The subspace spanned by  $s', \omega$  is invariant to  $U$ ; by denoting  $|\omega\rangle = |0\rangle$  and  $|s'\rangle = |1\rangle$ ,

$$\begin{aligned} U &= \frac{1}{N} \begin{pmatrix} N-2 & -2\sqrt{N-1} \\ 2\sqrt{N-1} & N-2 \end{pmatrix} = \mathbb{1} \cos\left(\frac{2\sqrt{N-1}}{N}\right) - i \sin\left(\frac{2\sqrt{N-1}}{N}\right) \sigma^y + O(N^{-3/2}) \\ &= e^{-2i\frac{\sqrt{N-1}}{N}\sigma^y} + O(N^{-3/2}). \end{aligned} \quad (49)$$

Here we used the following:

$$e^{i\varphi\sigma^y} = \sum_{j=0,1,2,\dots} \frac{(i\varphi\sigma^y)^j}{j!} = \sum_{j=0,2,4,\dots} \frac{(i\varphi)^j}{j!} \cdot \mathbb{1} + \sum_{j=1,3,5,\dots} \frac{(i\varphi)^j}{j!} \cdot \sigma^y = \mathbb{1} \cos(\varphi) + i\sigma^y \sin(\varphi) \quad (50)$$

$$1 - \frac{2}{N} + O(N^{-2}) = \cos\left(\frac{2\sqrt{N-1}}{N}\right) \quad (51)$$

$$\frac{2\sqrt{N-1}}{N} + O(N^{-3/2}) = \sin\left(\frac{2\sqrt{N-1}}{N}\right) \quad (52)$$

Denote  $H = 2\sigma^y/\sqrt{N}$ ; then  $H$  has eigenstates  $\frac{1}{\sqrt{2}}(|s'\rangle \pm i|\omega\rangle)$ , and additionally,

$$\|e^{-iH} - U\| = \left\| e^{-2i\sigma^y/\sqrt{N}} - e^{-2i\sigma^y\sqrt{N-1}/N} + O(N^{-3/2}) \right\| = O(N^{-1}) \quad (53)$$

Measuring an eigenstate of  $H$  in the original standard basis returns  $\omega$  with probability half. Thus an algorithm equivalent to Grover's is to apply an energy measurement of the state  $s$  with respect to the Hamiltonian  $H$ , with sufficient accuracy to arrive at a state close to an eigenstate, and then to measure in the original standard basis. Since the two eigenvalues differ by  $\theta(\frac{1}{\sqrt{N}})$ , it turns out that it suffices to perform a measurement with  $\eta$ -accuracy  $N^{-1/2}/10$  for  $\eta = 1 - 10^{-3}$  to achieve probability at least  $1/3$  to measure  $\omega$ . The exact argument follows from similar arguments to those in the proofs of claims 6 and 7<sup>4</sup>.

The energy measurement with such accuracy is implemented by phase estimation of  $U$  in complexity  $O(N^{-1/2})$  by choosing  $\ell = b + 10$ ,  $b = \lceil 4 + \frac{1}{2} \log N \rceil$  in Lemma 1. Note that  $U$  only approximates  $e^{-iH}$ . The phase estimation circuit (figure 1) uses  $O(\sqrt{N})$  instances of  $U$ , hence the state after the circuit (before the collapse) deviates by  $O(\sqrt{N}\|U - e^{-iH}\|) = O(N^{-1/2})$  compared to phase estimation of  $e^{-iH}$ . However the probability to measure  $\omega$  after this approximation is still greater than  $1/3$  in these parameters.

No FF is required for the speedup (compared to a classical algorithm), as this phase estimation only applies  $H$  for time durations which are at most  $O(\sqrt{N})$ . The quadratic speed-up is achieved by the mere fact that the accuracy required to separate the two eigenstates is of the order of  $1/\sqrt{N}$  and not  $1/N$ .

<sup>4</sup>Let  $\varepsilon_j, g$  be the energy measurement output and the state of the garbage register respectively, and let  $a_j$  be the amplitude of the eigenstate with energy closest to  $\varepsilon_j$ . According to claim 6,  $\mathbb{E}_{j,g}|a_j|^2 \geq \eta$ . Similarly to claim 7, given  $j, g$  the marked state is found with probability  $\Pr(\omega|j, g) = |a_j|^2 \left( \frac{1+99\sqrt{2}}{2} \right) - \frac{99}{\sqrt{2}} - 0.01$ , and thus  $\Pr(\omega) \geq \eta \left( \frac{1+99\sqrt{2}}{2} \right) - \frac{99}{\sqrt{2}} - 0.01$ .

3. **Exponentially fast solutions of linear equations [40, 41]:** Given an  $N \times N$  Hermitian  $s$ -row computable matrix  $A$ , and a state  $|b\rangle$ , the algorithm [40] finds the state  $|x\rangle = \sum_i x_i |i\rangle$  for  $x$  that solves the equation  $Ax = b$  (the non Hermitian case can be easily reduced to the Hermitian case). The time complexity of the algorithm is  $O(\text{poly}(\log(N), \kappa, 1/\epsilon))$ , where  $\kappa$  is the condition number of  $A$ , i.e., the ratio between the largest and smallest eigenvalues of  $A$ , and  $\epsilon$  is the additive error of  $|x\rangle$  allowed. The heart of the algorithm is a phase estimation of the unitary matrix  $e^{iA}$  applied to the state  $|b\rangle$ . The Hamiltonian simulation procedures used to simulate  $e^{-iAt}$  in [40, 41] apply for any  $A$ , thus both require at least linear computational complexity in  $t$ . If it weren't so, one could violate cTEUP for unknown Hamiltonians - contradicting Theorem 1. Hence no fast forwarding is involved.

As for other famous quantum algorithmic speed-ups, these do not seem to have a sensible description in terms of energy measurements of associated Hamiltonians, so they also do not seem to be related to FF. In particular, Kuperberg's sub-exponential algorithm for finding a hidden subgroup of the Dihedral group [42] and BQP-complete Topological Quantum Field Theory (TQFT) based quantum algorithms [43–45], do not seem to have a FF origin.

## 7.1 A note on Graph Automorphism

We consider two problems related to the symmetric group, or the group of permutations of  $n$  elements, denoted  $S_n$ .

**Definition 5. Permutation orbit:**

**Input :** Two permutations  $\sigma, \tau \in S_n$ .

**Output :** Is there a  $k$  such that  $\sigma^k = \tau$  (i.e., does the subgroup of  $S_n$  generated by  $\sigma$  contain  $\tau$ ?).

The above problem is phrased entirely in group theoretical language, and it turns out that it has a  $\text{poly}(n)$  classical solution<sup>5</sup>.

One can define a variant of this problem in which the permutation group acts on the set of graphs rather than on itself. We call this the Cyclic Graph Automorphism problem (CGA)

**Definition 6. Cyclic Graph Automorphism**

**Input :** An  $n$  vertex graph  $\Gamma$ , and a permutation  $\sigma \in S_n$ .

**Output :** Is there an automorphism  $\tau$  of  $\Gamma$  s.t.  $\tau = \sigma^k$  for some integer  $k$ . (i.e., does the subgroup of  $S_n$  generated by  $\sigma$  contain an automorphism of  $\Gamma$ ?).

We recall that a permutation  $\tau$  is an automorphism of a graph  $\Gamma(V, E)$  if

$$(v, u) \in E \Leftrightarrow (\tau(v), \tau(u)) \in E \quad (54)$$

This is a highly restricted version of the notoriously challenging graph automorphism problem (GA) (The general version, GA, is known to be at least as hard as the graph isomorphism problem [47], which is a long standing open problem for quantum algorithms; Though a recent breakthrough by Babai [48] gave a quasi-polynomial classical algorithm for the problem.)

We observe that unlike the Permutation Orbit problem (Definition 5), the related CGA problem seems like a hard problem for classical computers. In particular, the size of the largest order of an element of the symmetric group  $S_n$  is given by Landau's function  $g(n)$ , which is super polynomial:

$$\lim_{n \rightarrow \infty} \frac{\ln(g(n))}{\sqrt{n \ln(n)}} = 1. \quad (55)$$

Hence, the CGA problem cannot be solved in polynomial time by a brute-force search. Moreover, the classical algorithm for the group-theoretical variant of the problem, given in Footnote 5, cannot be applied to CGA since  $\tau$  is not given explicitly.

The following claim is easy to see:

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<sup>5</sup> We're to find the minimal  $k$  s.t.  $\tau = \sigma^k$ . For any  $j = 1..n$  we can exhaustively find the minimal  $0 < k_j < n$  s.t.  $\tau(j) = \sigma^{k_j}(j)$  (if no solution is found for any  $k_j$ , then  $\tau$  is not a power of  $\sigma$ ). Furthermore,  $\tau(j) = \sigma^{k_j + m r_j}(j)$  for every  $0 < m \in \mathbb{N}$ , where  $r_j$  is the size of the cycle in  $\sigma$  containing  $j$ . The problem reduces to finding  $k$  satisfying a system of linear congruences  $k \equiv k_j \pmod{r_j}$ . Solving such system is done by an efficient classical algorithm [46].

**Claim 9.** An efficient quantum algorithm exists for the CGA problem.

The proof is essentially by finding the length of the orbit as in Shor’s algorithm; this is due to the ability to fast forward the Hamiltonian corresponding to the application of  $\sigma$  on the given graph.

*Proof.* Let  $K$  be a cyclic subgroup of  $S_n$  of order at most  $g(n)$ , generated by  $\sigma$ . Let  $U_\sigma$  be the unitary that applies the group action of  $\sigma$  on a graph  $\Gamma$  (represented here by its adjacency matrix):

$$U_\sigma |\Gamma, \Gamma\rangle = |\Gamma, \sigma\Gamma\sigma^{-1}\rangle \quad (56)$$

If there is no automorphism of  $\Gamma$  in  $K$  then the orbit of  $\Gamma$  under the subgroup generated by  $\sigma$  is of size equal to the order of  $\sigma$  (namely the minimal  $r$  such that  $\sigma^r$  is the identity). Otherwise this orbit is of size  $m < r$  s.t.  $m$  is the minimal positive integer satisfying  $\sigma^m = \tau$  for  $\tau$  at automorphism. The order of  $\sigma$ ,  $r$ , can be calculated classically efficiently<sup>6</sup>. Suffices then to compute the order of  $\Gamma$  under the action of  $U_\sigma$  and compare it to  $r$ . This can be done using phase estimation for the unitary  $U_\sigma$  applied with the initial state  $|\Gamma, \Gamma\rangle$ , exactly as described in Section 3, using the fact that the permutation which is derived by taking an exponentially large power of  $\sigma$  can be calculated efficiently classically (by taking exponential power of each permutation cycle composing  $\sigma$ ).  $\square$

To the best of our knowledge, the fact that quantum computation is useful in *any* context of GI, albeit highly restricted, was not noticed before; This is probably because the Permutation Orbit problem, namely the group theoretical version of the CGA problem, is easy classically, and hence the problem was not considered.

Note that if we randomly pick  $\sigma \in S_n$ , the probability that  $\tau \in K$  is at most  $\frac{g(n)}{n!} \approx \frac{e^{\sqrt{n \ln n}}}{n!} = O(e^{-n \ln n + n + \sqrt{n \ln n} - \frac{\ln n}{2}})$ , which means that the above algorithm cannot be used to provide a algorithmic speed up for the GA or graph isomorphism problem.

We remark that the above algorithm can be easily extended to any Abelian subgroup of  $S_n$  (the largest such group is of size  $3^{n/3}$ ) using the algorithm for finding Abelian hidden subgroup [35].

## 8 Related work

### 8.1 Quantum metrology and the Heisenberg limit

The question we study here, of efficiently measuring energy of eigenstates, is different, though related, to the question of *distinguishing Hamiltonians* studied in [3], or to the related question of *sensing* in quantum metrology [7, 8].

A typical problem in quantum metrology is to estimate a parameter  $\gamma$  of a Hamiltonian  $H = \gamma H_0$  that resides in a black box, where  $H_0$  is a known dimensionless Hamiltonian. Consider a Mach-Zehnder interferometer aimed to determine the phase  $\gamma$  of a phase shifter (i.e.,  $H = \gamma$ , and the probe, namely the photon, passes through the box in one time unit.). Assuming the optical paths are equal, the Fock state of the probe before the measurement is  $\cos(\gamma/2) |1\rangle_a |0\rangle_b + i \sin(\gamma/2) |0\rangle_a |1\rangle_b$ , where  $a, b$  denote the two spatial modes.  $\gamma$  is estimated by measuring which of the paths the probe had taken (and of course taking statistics over many experiments). Classically, one can improve the accuracy (in this context, the standard deviation) by repeating the experiment  $n$  times. Assuming no correlation between probes, the accuracy is improved by a factor of  $\sqrt{n}$ . This is known as the *standard quantum limit*. When quantum correlations are introduced, the accuracy can bypass the standard quantum limit [49]. For instance, the system is in the a “Noon state” [50, 51]  $|n\rangle_a |0\rangle_b + |0\rangle_a |n\rangle_b$ , before entering the phase shifter. Each probe then gets a global phase factor as it passes through the phase shifter, and the state just before the measurement becomes  $\cos(n\gamma/2) |n\rangle_a |0\rangle_b + i \sin(n\gamma/2) |0\rangle_a |n\rangle_b$ . The distinguishability compared to a single probe is improved by a factor of  $n$ . When the Hamiltonian is accessed as a black box, this scaling of accuracy as  $1/n$  is optimal; it is referred to as the *Heisenberg limit* [7–10], where  $n$  is the number of times the Hamiltonian was applied (for one time unit each)<sup>7</sup>.

<sup>6</sup>Let  $\sigma = \prod_i \sigma_i$ , where  $\sigma_i$  are the disjoint permutation cycles composing  $\sigma$ . Let  $r_i$  be the order of  $\sigma_i$  ( $r_i$  is simply the length of the  $i^{\text{th}}$  cycle). The order of  $\sigma$ ,  $r = \text{lcm}(r_1, r_2, \dots)$

<sup>7</sup>A careful examination shows that no entanglement is necessary if one is allowed to modify the apparatus: one can direct a single photon  $n$  times through the phase shifter, and get exactly the same distribution as in the Noon state experiment

Such limits seem related to the TEUP; In particular, the metrology setting considers Hamiltonians which are promised to be of the form  $H = \gamma H_0$ , where  $H_0$  is known and  $\gamma$  is unknown; Inaccuracy in  $\gamma$  is thus directly related to inaccuracy in the energy eigenvalues. Moreover, the metrology setting allows accessing the Hamiltonian only through a black box. This is very reminiscent of the conditions for TEUP with unknown eigenvalues. It is thus natural to ask whether the Heisenberg limit is equivalent in some sense to the TEUP for Hamiltonians with unknown eigenvalues, Theorem 8. Indeed, it seems that an equivalence between the two should be provable, but we have not worked out the details. Roughly, the argument is as follows.

To derive a TEUP from the Heisenberg limit, assume we are given (or can generate) an eigenstate of  $H_0$  with eigenvalue  $E$ , and also assume we can apply an energy measurement with standard error  $\text{std}(E)$  on this eigenstate. This implies a standard error  $\text{std}(E)/E$  of  $\gamma$ ; However in this setting we can apply the Heisenberg limit (it holds even though we assume infinite computational power outside of the black boxes, so we can assume we can generate the eigenstate). This sets an upper bound on the accuracy to estimate  $\gamma$ , and so it entails a bound on the accuracy of the energy measurement. The details of this implications remain to be filled in, since the two settings use different accuracy and resource measures.

The other direction, of deriving a version of the Heisenberg limit from the TEUP, is a little less clear. Indeed, one can reduce the energy measurement problem to the problem of estimating  $\gamma$ , by applying the transformation  $|\psi_E, 0\rangle \mapsto |\psi_E, \gamma' E\rangle$ , with  $\gamma'$  being the estimated value of  $\gamma$ . However, we do not have a TEUP for the case of Hamiltonians that are completely known up to a scalar factor  $\gamma$ ; it seems that such a TEUP should follow from arguments similar to those in [2], but this remains to be done.

## 8.2 Other quantum bounds

### 8.2.1 Hamiltonian in a black box

We review two bounds on Hamiltonian dynamics.

The *Mandelstam-Tamm relation* [11, 52, 53], which was also named “the time energy uncertainty principle”, is the following:

$$\|\langle \psi | e^{-iHt} | \psi \rangle\|^2 \geq \cos^2 \left( \frac{\Delta E t}{\hbar} \right) \quad 0 \leq t \leq \pi \hbar / 2 \Delta E, \quad (57)$$

and  $(\Delta E)^2 = \langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2$ . Namely, the change of the quantum state’s direction, when evolving under the Hamiltonian  $H$ , is bounded from below in terms of its energy variance with respect to eigenvalues of  $H$ .

Let  $H$  be a Hamiltonian with ground energy 0. The *Margolus-Levitin bound* on the time duration required for a state  $\psi$  to evolve under  $H$  to a state orthogonal to  $\psi$  is the following:

$$\tau_{\perp} \geq \frac{\pi \hbar}{2 \langle \psi | H | \psi \rangle} \quad (58)$$

Here the state’s evolution is bounded by the expectation value of the energy.

Both bounds refer to the dynamics of a given Hamiltonian, which may not be altered (in a similar way to when  $H$  is applied as a black box). Hence these bounds do not apply to shortcuts such as fast forwarding, or simulations of the Hamiltonian by equivalent quantum circuits. A relation between the two bounds and the Heisenberg limit is discussed in [9].

### 8.2.2 The No-FF Theorem in the Query Model

A different model of Hamiltonian simulation, called the Hamiltonian query model, was studied in [18, 54–57]. In this model, the access to the entries of the row-sparse Hamiltonian is by queries to a row-oracle. Given the index of the row, the row-oracle returns the column numbers and the corresponding values of all non-zero elements in the row. Hamiltonian simulation algorithms in this model [18, 54–57], minimize the number of queries. Unlike Hamiltonians in which the circuit simulating the Hamiltonian is known (which is the subject of most of this paper) in the query model the circuit applying the Hamiltonian is *hidden* and therefore the only way to gain information about the Hamiltonian is by queries.

The most recent algorithm [18] for simulating a  $d$ -sparse normalized Hamiltonian  $H$  for time  $t$  with additive error  $\varepsilon$  in this query model requires  $O(\frac{\tau \log(\tau/\varepsilon)}{\log \log(\tau/\varepsilon)})$  queries, with  $\tau = d^2 t$ . Note that the query



complexity is more than linear in the time of the simulation, and it depends logarithmically on  $\varepsilon^{-1}$ . It is independent of the number of qubits.

In [54], it was shown that this behavior on time is essentially optimal: no procedure could simulate Hamiltonians in the query model for time  $t$  with query complexity that is sub-linear in  $t$ . This can be viewed as a no-fast-forwarding theorem for the query model. (The proof is by showing that such a procedure implies an algorithm for finding the parity of a binary string in less queries than the known lower bound [58]). One might wonder whether this theorem can be derived from the cTEUP for unknown Hamiltonians (Theorem 1 adapted from [2]) which together with our theorem 3 implies a no-FF for unknown Hamiltonians. However this result cannot be applied for the query model; the reason is that the query model has better distinguishability than the model of unknown Hamiltonians. For example, finding an entry of the Hamiltonian with accuracy of  $\text{poly}(n)$  bits costs one query in the query model, unlike the exponential time required in the case where the Hamiltonian is given as a black box.

We can summarize the comparison between the three models: a Hamiltonian given as a black box or one with unknown eigenvalues can't be Fast forwarded as this violates the TEUP/cTEUP for unknown Hamiltonians (Theorems 8,1 respectively). Adding information on the Hamiltonian by giving access to it by row oracle still won't allow a general FF procedure due to the no-FF theorem relying on query complexity bounds [54]. Our Theorem 6 is the corresponding theorem for the case of 2-sparse row computable Hamiltonians; Since we are no longer in the black box model, or even in the query model, we must condition the result on computational assumptions (here, we rely on the widely believed assumption that  $\text{PSPACE} \neq \text{BQP}$ ).

### 8.3 Susskind's complexification of a wormhole's length

Theorem 6 surprisingly relates to a recent conjecture by Susskind on the length of non-traversable wormholes .

The AdS/CFT correspondence [59] is a conjectured equivalence, or duality, between certain quantum gravity theories in anti de-Sitter (AdS) spacetime and Conformal Field theories (CFTs). The equivalence implies that any physical process can be formalized in either theory, and give the same predictions. An exact transformation between the two theories is yet to be found.

The CFT dual of a non-traversable wormhole is the maximally entangled state, which evolves in time under the transformation:

$$|\psi_t\rangle = 2^{-n/2} \sum_{y=1}^{2^n} |y\rangle \otimes U^t |y\rangle \quad U = V^{-1}V^T, \quad (59)$$

where  $V$  is a unitary. Susskind [13] has recently proposed the very intriguing suggestion, that the CFT dual of the length of non-traversable wormholes is equal to the *quantum circuit complexity* required to approximate  $\psi_t$ . This proposition is currently the only one producing the expected behaviour of non-traversable wormhole length in cases of interest. For instance, quantum gravity predicts that the length of the wormhole grows linearly in time [60] until a bound of  $2^n \text{poly}(n)$  is reached, and shrinks at time  $\approx 2^{2^n}$ . In terms of state complexity, by time  $t < 2^{2^n}$ ,  $\psi_t \approx \psi_0$  as can be proven by a counting argument; whereas  $2^n \text{poly}(n)$  is the upper bound on the complexity for approximating a state with constant error (generate one amplitude of a standard basis state at a time). A natural question is whether there exists a unitary  $U$  which achieves this upper bound, or in more picturesque words, *complexifies* the state  $\psi_t$  so rapidly. Aaronson and Susskind [12,61] prove the existence of a unitary with almost such qualities, under a commonly believed computational assumption ( $\text{PSPACE} \not\subseteq \text{PP/poly}$ ). More precisely, under this assumption, there exists a unitary  $U$ , and some  $t < 2^n$ , for which  $\psi_t$  (as in Equation 59) cannot be approximated by a polynomial size quantum circuit. The proof is of course just existential and says nothing about the specific unitary corresponding to the physical time evolution of the CFT; Susskind's conjecture is that the CFT Hamiltonian is indeed an example for such a highly "complexifying" evolution (see also Aaronson's notes [61]).

Our Theorem 6, and more generally the notion of fast forwarding, are very tightly connected to Aaronson and Susskind's result and to the question of complexification, and we now explain this connection. Aaronson and Susskind's theorem states (roughly) that there exists a  $U$  such that  $\psi_t$  cannot be approximated efficiently for some  $t < 2^n$ . A different way to say this is that the Hamiltonian  $H$  generating the unitary  $U$ , s.t.  $U = e^{-iH}$ , cannot be exponentially *fast-forwarded* - or else, the state  $\psi_t$

would be easy to generate efficiently by a quantum circuit simulating that Hamiltonian. In fact, what they show is even stronger - because they talk about approximating the state of Equation 59 which is an *average* over the time evolution under  $H$  of all basis states, they in fact rule out fast-forwarding even for just a close to 1 *fraction* of the states in the standard basis. This is stronger than what our theorem 6 implies; our theorem only rules out fast forwarding of *all* states. On the other hand, our Theorem 6 assumes less: our computational assumption is only  $\text{PSPACE} \neq \text{BQP}$  which is a weaker assumption (namely, more likely to hold). It is plausible that one can combine the two results to get a stronger claim with a weaker assumption; this is left for future work.

More generally, the notion of fast forwarding and limitations on it thus seem relevant also in the context of questions arising in quantum gravity.

## 9 Conclusions and Open Questions

A fundamental open question remains: Is it possible to characterize, by Physical terms, the exact conditions for fast forwarding? Or equivalently, for super-efficient energy measurements? What is the true physical source for such a possibility?

A very intriguing question is whether many body localization Hamiltonians, which had attracted much attention recently [62], and which are in some sense a generalization of commuting local Hamiltonians, can be fast forwarded efficiently; by Theorem 3 this would imply the ability to measure the energy of such systems with exponential accuracy.

The notion of fast forwarding Hamiltonians, namely, projecting the physical system far into the future using computationally limited resources, captures the imagination. What else can be done with such systems, experimentally or theoretically?

The relation between quantum algorithms and fast forwarded Hamiltonian put forward in this work also raises a question motivated by computer science - what other interesting problems can have their solution hidden in the exponentially accurate spectrum of a Hamiltonian that can be fast forwarded? Could finding new examples for fast forwarding interesting Hamiltonians, lead to new quantum algorithms?

We finally mention that our work leaves open the question of whether parameter estimation in sensing and metrology can also be dramatically improved (beyond the Heisenberg limit, as in [63,64]). As explained in 8.1, this is impossible to do in the usual sensing paradigm in which the Hamiltonian is given in a black box. However it is conceivable that one can use more knowledge about the Hamiltonian and possibly quantum computational techniques such as fast forwarding or others to go beyond the Heisenberg limit. This possibility is left for future exploration.

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## Supplemental material

### A TEUP for unknown Hamiltonians

We slightly strengthen the TEUP for completely unknown Hamiltonians [2], by showing it holds even when the eigenstates are known. Then we adapt it to comply with  $2/3$ -accuracy (definition 2) instead of mean deviation. In this section, our definition of a Hamiltonian with unknown eigenvalues is as stated in Theorem 1.

**Theorem 7** (TEUP for Hamiltonians with unknown eigenvalues (adapted from [2])). *If the eigenvalues of a Hamiltonian acting on a system are unknown, then the precision  $\Delta E$  with which one can estimate the energy of an eigenstate with energy  $E$  in a time obeys the constraint*

$$\Delta E \Delta t > 1/4. \quad (60)$$

$\Delta E$  is the mean deviation of the measurement:

$$\Delta E = \sum_{E'} \Pr(E'|E) |E - E'| \quad (61)$$

$\Delta t$  is the total time the Hamiltonian was applied.

*Proof.* The proof of TEUP for completely unknown Hamiltonians [2] holds for the case in which only eigenvalues are unknown, with  $\Delta t$  denoting the duration of the measurement, and the scheme is sequential, i.e., only one system is affected by the Hamiltonian at any given time. One might be worried however that by applying the Hamiltonian on several probes in parallel on entangled states as is done for example in the case of Noon states (see Section 8.1), one might be able to bypass the bound achieved in [2]. However, notice that given any measurement scheme which applies the Hamiltonian on several probes (registers) in parallel, one can apply standard quantum computation techniques of adding a register and swapping between registers, to arrive at an equivalent protocol which only applies the Hamiltonian on one probe (register) sequentially. Hence we take  $\Delta t$  to be the total time the Hamiltonian was applied.  $\square$

In terms of  $2/3$ -accuracy, Theorem 7 takes the following form:

**Theorem 8.** *Let  $H$  be a Hamiltonian with unknown eigenvalues. The  $2/3$ -accuracy  $\delta E$  of measuring the energy of an eigenstate depends on the time the Hamiltonian was sampled  $\Delta t$  by*

$$\delta E \Delta t \geq 1/3 \quad (62)$$

*Proof.* Assume by contradiction that there exists some family of Hamiltonians with unknown eigenvalues (but fixed eigenstates which are common to all), and also that there exists a given eigenstate and constants  $\delta E$  and  $\Delta t$ , s.t. one can perform an energy measurement of the eigenstate with  $2/3$ -accuracy  $\delta E$ , while applying the Hamiltonian for  $\Delta t$ , and yet  $\delta E \Delta t < \frac{1}{3}$ . We will derive a contradiction by showing that this implies a protocol which is too strong, for the distinguishability problem studied in [2].

The distinguishability problem is defined as follows: Given access to a Hamiltonian by a black box, determine whether the Hamiltonian in the box is  $H_1$  or  $H_2 = H_1 + \varepsilon \mathbb{1}$  (it is promised that the Hamiltonian in the box is one of the two, and it is assumed that there are no computational bounds outside the box, and in particular, we can feed the box any eigenstate we want). Both Hamiltonians have an a-priori probability  $1/2$ . Define the probability of error for a protocol for this task by:

$$P_{err} = \frac{1}{2} [\Pr(\text{output } 2|H_1) + \Pr(\text{output } 1|H_2)]. \quad (63)$$

Using our assumed energy measurement, we can derive a protocol for this distinguishability task between two Hamiltonians from the family:  $H_1$  and  $H_2 = H_1 + \varepsilon \mathbb{1}$ , with  $\varepsilon = 2/3\Delta t$ . Apply an energy measurement with  $2/3$ -accuracy  $\delta E < 1/3\Delta t = \varepsilon/2$ , to an eigenstate of the Hamiltonians (which by assumption we can generate). We know the energy of the eigenstate is either  $E$  or  $E + \varepsilon$ . The procedure outputs  $H_1$  if the measurement outcome is closer to  $E$  than to  $E + \varepsilon$  and outputs  $H_2$  otherwise. From the definition of  $2/3$ -accuracy, in this procedure,

$$P_{err} < 1/3. \quad (64)$$

However, one of the intermediate results on Hamiltonian distinguishability in [2] is the following:

**Lemma 4** ( $H$  distinguishability, adapted from [2] section III.B ). Any algorithm solving the distinguishability problem defined above for distinguishing between  $H_1$  and  $H_2 = H_1 + \varepsilon \mathbb{1}$ , while applying the Hamiltonian in the black box for a total time  $\Delta t$ , satisfies

$$P_{err} \geq \frac{1}{2} \left[ 1 - \sin \left( \frac{\varepsilon \Delta t}{2} \right) \right] \quad (65)$$

if  $\varepsilon \Delta t < \pi$ .

Combining this lemma and Equation 64 we have

$$\frac{1}{3} > P_{err} \geq \frac{1}{2} \left[ 1 - \sin \left( \frac{1}{3} \right) \right], \quad (66)$$

which is a contradiction.  $\square$

## B Proofs of claims in Section 5.2

**Claim 3.** Let  $\mathbf{a}$  be a column vector whose  $j^{\text{th}}$  coordinate is  $a_j$  and let  $\mathbf{a}^\dagger$  be a column vector whose  $j^{\text{th}}$  coordinate is  $a_j^\dagger$ . The Hamiltonian  $H$  can be written as

$$H = \frac{1}{2} \begin{pmatrix} \overline{\mathbf{a}^\dagger} & \overline{\mathbf{a}} \end{pmatrix} \begin{pmatrix} A & B^* \\ B & -A^* \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix} + \frac{1}{2} \text{tr}(A) \quad (67)$$

Here, the overline indicates a matrix transposition, i.e.,  $\overline{\mathbf{a}^\dagger}, \overline{\mathbf{a}}$  are the row vectors corresponding to  $\mathbf{a}^\dagger, \mathbf{a}$  respectively.

*Proof.* The proof relies on the hermiticity of  $A, B$  and the anticommutation relations. For  $i \neq j$

$$\sum_{i \neq j} A_{i,j} a_i^\dagger a_j = \frac{1}{2} \sum_{i \neq j} (A_{i,j} a_i^\dagger a_j - A_{i,j} a_j a_i^\dagger) = \frac{1}{2} \left( \sum_{i \neq j} A_{i,j} a_i^\dagger a_j - \sum_{j \neq i} A_{j,i} a_i a_j^\dagger \right) = \frac{1}{2} \sum_{i \neq j} (A_{i,j} a_i^\dagger a_j - A_{i,j}^* a_i a_j^\dagger) \quad (68)$$

For  $i = j$ ,

$$\sum_i A_{i,i} a_i^\dagger a_i = \frac{1}{2} \sum_i A_{i,i} a_i^\dagger a_i + \frac{1}{2} \sum_i A_{i,i} (1 - a_i a_i^\dagger) = \frac{1}{2} \left( \sum_i A_{i,i} a_i^\dagger a_i - \sum_i A_{i,i}^* a_i a_i^\dagger + \text{tr}(A) \right) \quad (69)$$

Reorganizing  $H$  as a block matrix concludes the proof.  $\square$

**Claim 4.** The traceless part of the Hamiltonian can be diagonalized:

$$H = \frac{1}{2} \begin{pmatrix} \overline{\mathbf{a}^\dagger} & \overline{\mathbf{a}} \end{pmatrix} U D U^\dagger \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix} + \frac{1}{2} \text{tr}(A) \quad (70)$$

where  $D$  is a real diagonal matrix s.t.  $D_{j,j} = -D_{j+n,j+n}$  and  $U$  is unitary. Furthermore, there exist matrices  $V_1, V_2$  s.t.  $U$  is a block matrix in the form  $U = \begin{pmatrix} V_1 & V_2^* \\ V_2 & V_1^* \end{pmatrix}$

*Proof.* Note the following symmetry of the traceless matrix  $\mathcal{H}$ :

$$\mathcal{H} \equiv \begin{pmatrix} A & B^\dagger \\ B & -A^* \end{pmatrix} \quad \mathcal{H}^\dagger = \mathcal{H}, \quad \tau \mathcal{H} \tau = -\mathcal{H}^*, \quad \tau \equiv \begin{pmatrix} 0 & \mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix} \quad (71)$$

The symmetry of  $\mathcal{H}$  implies a symmetry on its eigenvectors, which are the column vectors of  $U$ .

$$\mathcal{H} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \mathcal{H} \begin{pmatrix} v_2^* \\ v_1^* \end{pmatrix} = -\lambda \begin{pmatrix} v_2^* \\ v_1^* \end{pmatrix} \quad (72)$$

Hence if either  $v_1 \neq v_2^*$  or  $v_2 \neq v_1^*$ , then the eigenvectors come in pairs:  $w, \tau w^*$ . In the case  $v_1 = v_2^*$  and  $v_2 = v_1^*$ , we get that  $\lambda = 0$ . Since there is an even number of eigenvectors of the first case, and the dimension of the subspace is  $2m$ , there's also an even number of eigenvectors of the second case, with eigenvalue 0. Picking a pair of orthogonal eigenvectors  $w_1, w_2$ , with eigenvalue 0, we can span the subspace they define using  $w'_1 = w_1 + i w_2$  and  $w'_2 = w_1 - i w_2$ , and we get that  $w'_1 = \tau w'^*_2$ . Hence all eigenvectors of  $\mathcal{H}$  come in pairs  $w, \tau w^*$ . By placing one eigenvector of the  $i^{\text{th}}$  pair in column  $i$  of  $U$  and the other eigenvector in column  $m + i$ ,  $U$  takes the desired form. Additionally, the column placement forces  $D_{j,j} = -D_{j+m,j+m}$ .  $\square$

**Claim 5.** By defining

$$\begin{pmatrix} \mathbf{b} \\ \mathbf{b}^\dagger \end{pmatrix} = U^\dagger \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}. \quad (73)$$

The Hamiltonian takes the form

$$H = 2 \sum_{i=1}^m b_i^\dagger b_i D_{i,i} + \frac{1}{2} \text{tr}(A) \quad (74)$$

The new operators obey the anti-commutation relations of fermions. In addition, the eigenvalues of  $b_i^\dagger b_i$  are either 0 or 1.

*Proof.* From equations 73 and the definition of  $U$ ,

$$b_i = \sum_j (V_1^\dagger)_{i,j} a_j + (V_2^\dagger)_{i,j} a_j^\dagger \quad (75)$$

$$b_i^\dagger = \sum_j (\bar{V}_2)_{i,j} a_j + (\bar{V}_1)_{i,j} a_j^\dagger \quad (76)$$

Hence  $b_i^\dagger$  is indeed the hermitian conjugate of  $b_i$ . Additionally, one can see that  $\begin{pmatrix} \bar{\mathbf{b}}^\dagger & \bar{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{a}}^\dagger & \bar{\mathbf{a}} \end{pmatrix} U$ . Next we show that the transformation is canonical, i.e., the anticommutation relations are preserved:

$$\begin{aligned} \{b_i, b_j\} &= \left\{ \sum_{k=1}^m U_{k,i}^* a_k + \sum_{k=1}^m U_{k+m,i}^* a_k^\dagger, \sum_{\ell=1}^m U_{\ell,j} a_\ell + \sum_{\ell=1}^m U_{\ell+m,j} a_\ell^\dagger \right\} \\ &= \sum_{k=1}^m (U_{k,i}^* U_{k+m,j} + U_{k+m,i}^* U_{k,j}) = \sum_{k=1}^m (U_{k,i}^* U_{k,j+m} + U_{k+m,i}^* U_{k+m,j+m}) = 0 \end{aligned} \quad (77)$$

(we used the structure of  $U$  for the transition of the second line)

$$\{b_i, b_j^\dagger\} = \left\{ \sum_{k=1}^m U_{k,i}^* a_k + \sum_{k=1}^m U_{k+m,i}^* a_k^\dagger, \sum_{\ell=1}^m U_{\ell,j} a_\ell^\dagger + \sum_{\ell=1}^m U_{\ell+m,j} a_\ell \right\} = \sum_{k=1}^m (U_{k,i}^* U_{k,j} + U_{k+m,i}^* U_{k+m,j}) = \delta_{i,j} \quad (78)$$

Finally, The eigenvalues of  $b_j^\dagger b_j$  are 0 and 1, because  $(b_j^\dagger b_j)^2 = b_j^\dagger b_j b_j^\dagger b_j = (1 - b_j b_j^\dagger) b_j^\dagger b_j = b_j^\dagger b_j$ , similarly to the number operator  $a_j^\dagger a_j$ .  $\square$

## References

- [1] Y. Aharonov and D. Bohm. Time in the quantum theory and the uncertainty relation for time and energy. *Physical Review*, 122(5):1649–1658, 1961.
- [2] Y. Aharonov, S. Massar, and S. Popescu. Measuring energy, estimating hamiltonians, and the time-energy uncertainty relation. *Phys. Rev. A: At., Mol., Opt. Phys.*, 66:052107, 2002.
- [3] Andrew M Childs, John Preskill, and Joseph Renes. Quantum information and precision measurement. *Journal of modern optics*, 47(2-3):155–176, 2000.
- [4] Peter W. Shor. Algorithms for quantum computation: discrete logarithms and factoring. In *Proceedings of the 35th Annual Symposium on Fundamentals of Computer Science (FOCS'94)*, volume 35, pages 124–134, 1994.
- [5] Asher Peres. *Quantum Theory: Concepts and Methods*. Fundamental Theories of Physics. Kluwer Academic Publishers, Dordrecht, 1993.
- [6] Paul Busch. The time-energy uncertainty relation. In *Time in quantum mechanics*, pages 73–105. Springer, 2008.
- [7] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Quantum metrology. *Physical review letters*, 96(1):010401, 2006.
- [8] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Advances in quantum metrology. *Nature Photonics*, 5(4):222–229, 2011.
- [9] Marcin Zwierz, Carlos A Pérez-Delgado, and Pieter Kok. General optimality of the heisenberg limit for quantum metrology. *Physical review letters*, 105(18):180402, 2010.
- [10] Marcin Zwierz, Carlos A Pérez-Delgado, and Pieter Kok. Ultimate limits to quantum metrology and the meaning of the heisenberg limit. *Physical Review A*, 85(4):042112, 2012.

- [11] L Mandelstam and Igor Tamm. The uncertainty relation between energy and time in nonrelativistic quantum mechanics. *J. Phys.(USSR)*, 9(249):1, 1945.
- [12] Scott Aaronson and L. Susskind. In prepatation.
- [13] Leonard Susskind. Computational complexity and black hole horizons. *Fortschritte der Physik*, 64(1):24–43, 2016.
- [14] Ethan Bernstein and Umesh Vazirani. Quantum complexity theory<sup>†</sup>. *SIAM J. Comp.*, 26(5):1411–1473, 1997.
- [15] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
- [16] Vladimir B Braginsky, Yuri I Vorontsov, and Kip S Thorne. Quantum nondemolition measurements. *Science*, 209(4456):547–557, 1980.
- [17] Dorit Aharonov and Amnon Ta-Shma. Adiabatic quantum state generation and statistical zero knowledge. In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, pages 20–29. ACM, 2003.
- [18] Dominic W Berry, Andrew M Childs, Richard Cleve, Robin Kothari, and Rolando D Somma. Exponential improvement in precision for simulating sparse hamiltonians. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 283–292. ACM, 2014.
- [19] Joseph Rotman. *An introduction to the theory of groups*, volume 148. Springer Science & Business Media, 2012.
- [20] Peter W. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM J. Comp.*, 26(5):1484–1509, 1997.
- [21] A Yu Kitaev. Quantum measurements and the abelian stabilizer problem. *arXiv preprint quant-ph/9511026*, 1995.
- [22] Victor Y Pan and Zhao Q Chen. The complexity of the matrix eigenproblem. In *Proceedings of the thirty-first annual ACM symposium on Theory of computing*, pages 507–516. ACM, 1999.
- [23] Diego Armentano, Carlos Beltrán, Peter Bürgisser, Felipe Cucker, and Michael Shub. A stable, polynomial-time algorithm for the eigenpair problem. *arXiv preprint arXiv:1505.03290*, 2015.
- [24] A Yu Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2–30, 2003.
- [25] Michael J Bremner, Richard Jozsa, and Dan J Shepherd. Classical simulation of commuting quantum computations implies collapse of the polynomial hierarchy. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*. The Royal Society, 2010.
- [26] Michael J Bremner, Ashley Montanaro, and Dan J Shepherd. Achieving quantum supremacy with sparse and noisy commuting quantum computations. *arXiv preprint arXiv:1610.01808*, 2016.
- [27] Dorit Aharonov and Michael Ben-Or. Fault-tolerant quantum computation with constant error. In *Proc. 29<sup>th</sup> STOC*, pages 176–188. ACM, 1997.
- [28] Emanuel Knill, Raymond Laflamme, and Wojciech H. Zurek. Resilient quantum computation. *Science*, 279(5349):342–345, 1998.
- [29] Daniel Gottesman. An introduction to quantum error correction and fault-tolerant quantum computation. *Quantum Information Science and Its Contributions to Mathematics*, 68:13–60, 2009.
- [30] P. W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109:1492–1505, Mar 1958.
- [31] Claude Itzykson and Jean-Bernard Zuber. *Quantum field theory*. Courier Corporation, 2006.



- [32] Jean-Paul Blaizot and Georges Ripka. *Quantum theory of finite systems*, volume 3. Mit Press Cambridge, 1986.
- [33] VS Shchesnovich. The second quantization method for indistinguishable particles (lecture notes in physics, ufabc 2010). *arXiv preprint arXiv:1308.3275*, 2013.
- [34] Christos H Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *Journal of Computer and system Sciences*, 48(3):498–532, 1994.
- [35] Michele Mosca and Artur Ekert. The hidden subgroup problem and eigenvalue estimation on a quantum computer. In *Quantum Computing and Quantum Communications*, pages 174–188. Springer, 1999.
- [36] Sean Hallgren. Polynomial-time quantum algorithms for pell’s equation and the principal ideal problem. *Journal of the ACM (JACM)*, 54(1):4, 2007.
- [37] Kirsten Eisenträger, Sean Hallgren, Alexei Kitaev, and Fang Song. A quantum algorithm for computing the unit group of an arbitrary degree number field. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 293–302. ACM, 2014.
- [38] A .M. Childs, R. Cleve, E. Deotto, E. Farhi, S. Gutmann, and D .A. Spielman. Exponential algorithmic speedup by a quantum walk. In *Proceedings of the Thirty-fifth Annual ACM Symposium on Theory of Computing*, STOC ’03, pages 59–68, New York, NY, USA, 2003. ACM.
- [39] Lov K. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing*, pages 212–219, New York, 22–24 May 1996.
- [40] Aram W Harrow, Avinatan Hassidim, and Seth Lloyd. Quantum algorithm for linear systems of equations. *Physical review letters*, 103(15):150502, 2009.
- [41] Andrew M Childs, Robin Kothari, and Rolando D Somma. Quantum linear systems algorithm with exponentially improved dependence on precision. *arXiv preprint arXiv:1511.02306*, 2015.
- [42] Greg Kuperberg. A subexponential-time quantum algorithm for the dihedral hidden subgroup problem. *SIAM Journal on Computing*, 35(1):170–188, 2005.
- [43] Michael H Freedman, Alexei Kitaev, and Zhenghan Wang. Simulation of topological field theories by quantum computers. *Communications in Mathematical Physics*, 227(3):587–603, 2002.
- [44] M Bordewich, M Freedman, L Lovász, and D Welsh. Approximate counting and quantum computation. *Combinatorics, Probability and Computing*, 14(5-6):737–754, 2005.
- [45] Dorit Aharonov, Vaughan Jones, and Zeph Landau. A polynomial quantum algorithm for approximating the jones polynomial. *Algorithmica*, 55(3):395–421, 2009.
- [46] Kenneth H Rosen. *Elementary Number Theory and its Applications*, 2005. Addison Wesley.
- [47] Johannes Köbler, Uwe Schöning, and Jacobo Torán. *The graph isomorphism problem: its structural complexity*. Birkhauser Verlag, 1994.
- [48] László Babai. Graph isomorphism in quasipolynomial time [extended abstract]. In *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2016, pages 684–697, New York, NY, USA, 2016. ACM.
- [49] B Yurke. Input states for enhancement of fermion interferometer sensitivity. *Physical review letters*, 56(15):1515, 1986.
- [50] Barry C Sanders. Quantum dynamics of the nonlinear rotator and the effects of continual spin measurement. *Physical Review A*, 40(5):2417, 1989.
- [51] Hwang Lee, Pieter Kok, and Jonathan P Dowling. A quantum rosetta stone for interferometry. *Journal of Modern Optics*, 49(14-15):2325–2338, 2002.

- [52] Gordon N Fleming. A unitarity bound on the evolution of nonstationary states. *Il Nuovo Cimento A (1971-1996)*, 16(2):232–240, 1973.
- [53] Kamal Bhattacharyya. Quantum decay and the mandelstam-tamm-energy inequality. *Journal of Physics A: Mathematical and General*, 16(13):2993, 1983.
- [54] Dominic W Berry, Graeme Ahokas, Richard Cleve, and Barry C Sanders. Efficient quantum algorithms for simulating sparse hamiltonians. *Communications in Mathematical Physics*, 270(2):359–371, 2007.
- [55] Andrew Macgregor Childs. *Quantum information processing in continuous time*. PhD thesis, Massachusetts Institute of Technology, 2004.
- [56] Andrew M Childs and Robin Kothari. Simulating sparse hamiltonians with star decompositions. In *Theory of Quantum Computation, Communication, and Cryptography*, pages 94–103. Springer, 2011.
- [57] Andrew M Childs and Nathan Wiebe. Hamiltonian simulation using linear combinations of unitary operations. *arXiv preprint arXiv:1202.5822*, 2012.
- [58] Robert Beals, Harry Buhrman, Richard Cleve, Michele Mosca, and Roland de Wolf. Quantum lower bounds by polynomials. In *Proceedings of the 39th IEEE Conference on Foundations of Computer Science*, pages 352–361, 1998.
- [59] Juan Maldacena. The large-n limit of superconformal field theories and supergravity. *International journal of theoretical physics*, 38(4):1113–1133, 1999.
- [60] Robert W Fuller and John A Wheeler. Causality and multiply connected space-time. *Physical Review*, 128(2):919, 1962.
- [61] Scott Aaronson. The complexity of quantum states and transformations: From quantum money to black holes. *arXiv preprint arXiv:1607.05256*, 2016.
- [62] Rahul Nandkishore and David A Huse. Many body localization and thermalization in quantum statistical mechanics. *arXiv preprint arXiv:1404.0686*, 2014.
- [63] David A Herrera-Martí, Tuvia Gefen, Dorit Aharonov, Nadav Katz, and Alex Retzker. Quantum error-correction-enhanced magnetometer overcoming the limit imposed by relaxation. *Physical review letters*, 115(20):200501, 2015.
- [64] Eric M Kessler, Igor Lovchinsky, Alexander O Sushkov, and Mikhail D Lukin. Quantum error correction for metrology. *Physical review letters*, 112(15):150802, 2014.